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Growth, Fiscal Policy and the Informal Sector in a Small Open Economy

Gui Pedro de Mendonça[1,2]

Abstract: We discuss the implications of informality on growth and fiscal policy by considering an informal sector based on low tech firms, in an open economy model of endogenous growth, where labour supply is elastic and increasing returns arise from public spending. We allow for both labour and capital to allocate between sectors and examine the dynamic and policy issues that arise in an economy, where long run outcomes are still dominated by formal activities, but long macroeconomic transitions arise as a result of informal microeconomic activities, which take advantage of both government taxation and limited fiscalization.

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[1]This research paper was based on one of the chapters from my Master's dissertation in Economics at the School of Economics and Management (ISEG), Technical University of Lisbon (UTL), under the supervision of Professor Miguel St. Aubyn (ISEG/UTL). This proposal was devised while waiting for a final decision on my dissertation and benefited from both insightful comments and suggestions by Miguel St. Aubyn. Miguel St. Aubyn is a researcher at the Research Unity on Complexity and Economics (UECE/ISEG/UTL). Original draft from March 2008, this revised version prepared for presentation at the above mentioned conference.

[2]PhD candidate in Complexity Sciences (2008-2012) at the Centre for Complexity Science, University of Warwick. I gratefully acknowledge financial support (January 2009 onwards) from the (Portuguese Government), Fundação para a Ciência e a Tecnologia, under the grant SFRH/BD/46267/2008.

1. Introduction

Ever since the introduction of endogenous growth theory and the increasing returns hypothesis, in the notable Romer (1986) article, theoretical growth economics has debated on the sources of increasing returns and the policy implications of endogenous growth. Earlier contributions by Lucas (1990), Barro (1990), Rebelo (1991), Barro and Sala-i-Martin (1992), Saint-Paul (1992) and Jones, Manuelli and Rossi (1993), suggested that government spending and taxing policies should have an important role on long run growth outcomes. We follow closely this set of proposals and develop a multiple instruments fiscal policy model, based on the Turnovsky (1999) proposal, of an open economy model with an elastic labour supply, where increasing returns arise from productive public spending, to discuss the implications of informality in the outcomes of long run growth and government fiscal policy.

Other recent extensions on the subject of government policy and endogenous growth, which follow similar modelling assumptions as ours, include fiscal policy models, where government spending is used, not only to provide public services, but also to invest in human capital formation, such as Ortigueira (1998) and Agénor (2005). Park and Philippopoulos (2003) discuss optimal fiscal policy and dynamic determinacy in a continuous deterministic endogenous growth model with increasing returns, generated by public infrastructure and additional non productive public spending, specifically public consumption services and redistributive transfers. Still, there is a large scope for discussing policy implications in endogenous growth theory. Our proposal serves this purpose by extending fiscal policy implications when tradeoffs arise from informality, in both capital and labour markets.

The work of Jones and Manuelli (1990) on convex models of endogenous growth provided a framework for introducing long transitions in endogenous growth models. Their proposal consisted in assuming that production was given separately, by two economic sectors with different production technologies, during transitions to the long run. One of the sectors produced macro outcomes with increasing returns, while the other followed neoclassical assumptions. The dynamic outcome of the original hypothesis consisted on a long run growth equilibrium determined by the increasing returns sector and saddle path transitions influenced by the neoclassical sector dynamical decay. Despite providing a simple framework for tackling both, two sector models of endogenous growth and introducing transitional dynamics, this proposal was somehow undermined by the long, rigid transitions arising both from convergence and comparative dynamic analysis. Industrial change and structural adjustment was not as persistent and long as this proposal suggested and, therefore, the Jones and Manuelli (1990) framework was dismissed as a reasonable approach to tackle these issues. Still, the potential to tackle persistent low tech industrial phenomena, subject to long adjustment periods was there and matched the observed outcomes of informality in developed countries. In Mendonça (2007) [3], we propose that the formal vs. informal outcomes for growth in a developed open economy may be portrayed in the Jones and Manuelli (1990) fashion. This hypothesis departs from some specific micro and macro assumptions. First, opportunities arise for entrepreneurial informal activities at the micro level, due to government lack of regulation and fiscalization, which can be shown to be consistent with long transitions at the macro level. Second, if we consider the informal sector of the economy to be accountable for long transitions towards long run growth optimal equilibrium, then matching continuous persistence of low tech informal activities can be shown to arise when innovations are considered during transitions. In this framework, informal activities would persist in a sub-optimal continuous adjustment process, where long run outcomes are not be given by a stable steady-state equilibrium, but are a sum of short to medium run periods of transitions.

2. Overview

Why is informality both important for growth and policy even in developed countries? Although, this is a current and important topic of development economics research, it has not been considered as a relevant topic in growth theory. This paradigm is based on a stylized fact, which considers that only the large dominant informal sectors of developing countries can be accounted to have both growth and policy implications. Where, in industrialized economies the relevance of this activity is limited and thus negligible. This stylized assumption may have been true for some advanced economies during the post war period, but is certainly not true nowadays. We start defying this assumption by reproducing bellow a table that briefly summarizes informal activities into straightforward categories:

Type of Activity	**Monetary Transactions**		**Nonmonetary Transactions**	
ILLEGAL ACTIVITIES	Trade in stolen goods; drug dealing and manufacturing; prostitution; gambling; smuggling, and fraud		Barter: drugs, stolen goods, smuggling etc. Produce or growing drugs for own use. Theft for own use.	
	Tax Evasion	**Tax Avoidance**	**Tax Evasion**	**Tax Avoidance**
LEGAL ACTIVITIES	Unreported income from self-employment; Wages, salaries and assets from unreported work related to legal services and goods	Employee discounts, fringe benefits	Barter of legal services and goods	All do-it-yourself work and neighbour help

Table 1- Taxonomy of Types of Underground Economic Activities[4]

[3] The basic framework for this model was first introduced in one of the chapters from master dissertation at ISEG/UTL. This paper extends both the analytical and numerical proposals for the specific case without investment adjustment costs.

[4] Reproduced from Schneider and Enste (2000).

From all definitions portrayed in table 1, there exists one important characteristic, which is common between all these specific classifications for informality. Each one of us, residents of industrialized countries, has already been, at least, partially exposed to these shadow activities and was given descriptions of these schemes, through third party information, ranging from informal talks to mass media coverage of this phenomenon. The straightforward conclusion for this widespread individual exposition to information on these activities is that they are relevant, persistent and widespread in social economic systems. The differences in both economic scale and diversity of these categories are also evidence that opportunities for profiteering from informality are diverse and not restricted to specific social and economic conditions. They are in fact an emergent macro outcome of social economic systems, arising from agent's behaviour and microeconomic market conditions. One straightforward conclusion to be drawn is that underground activities, which exhibit characteristics such as persistency, scale and diversity, must have both implications for growth and policy, even in developed economies.

The second hypothesis supporting the stylized assumptions about the implications of informality in industrialized nations, consists on the perceived correlation between informal business scale and economic relevance. The paradigm suggests that if the informal sector is small, then its impact must also be small, even if it persists in the long run. Research on the size and causes of informality contradicts this view for industrialized economies and provides evidence that this phenomenon is not only persistent in developed economies, but has also increased in dimension, during the past two decades, when compared to the relative formal sector dimension on total output share. Schneider (2005) reports an average size for the informal sector relative to GDP of 13,2% in 21 OECD economies, for the years 1989/90, using the currency demand and DYMIMIC method[5]. These values increase to an average size of 16,4% for the years 2002/03, representing an increase on total output share of about 24% for informal activities, during this recent thirteen years time frame, which represents an average annual growth of about 1,6% for the output share of informal activities. If we extend our time frame some decades to the past, using Frey and Schneider (2001) results, we can emphasize the increasing role of informality in developed economies. They report that Nordic economies, except for Finland, had an average informal sector size of less than 5% relative to GNP in 1960, according to the currency demand method. This output share increases to values of about 15% to 20%, when we consider the year 1995. The growth of output share for informal activities in industrialized economies varies according to the countries we consider, but we can

[5]Dynamic multiple-indicators multiple-causes method. Refer to Schneider (2005) for further details in this recent modelling technique.

state, with some degree of confidence, that an increasing share of output has been located in the informal sectors of industrialized economies during the past two decades. Moreover, we can also state that this specific sector is no longer residual and in some cases, as in south European countries, it represents between one fifth to almost one third of total output share.

What are the causes for this clear trend in developed economies and what are the consequences of this increasing output share for policy outcomes? There are two key causes that may explain this evidence and at the same time are able to capture this trend, even if we consider the specific socio-cultural and political backgrounds of different developed economies. These causes are increasing government spending and bureaucracy, in industrialized countries, during the past four decades. Both these causes are widely accepted for explaining this trend, because they match basic industrial economics theory. Increasing bureaucracy and taxes augment both fixed and marginal costs, distort market prices, consequently, creating legal and economic barriers to entry in the formal sector, decreasing present and future profits for formal sector firms and widening the range of opportunities for illegal activities. We try to tackle these two issues in an endogenous framework for a small developed economy, where informal activities arise during transitions, taking advantage of these arbitrage conditions at the micro level. Peñalosa and Turnovsky (2004) explore these same sources of opportunities for informal activities in a developing economy, considering a *CES* production function and increasing returns arising from the usual aggregate capital hypothesis. Although, our framework has many similarities with the hypotheses proposed in Peñalosa and Turnovsky (2004), the main scope of their article are the second optimal policies that arise in the presence of a government that lacks revenues and only has access to a limited formal sector for taxation. Other examples of articles that follow similar hypotheses for developing economies in an endogenous growth framework are Braun and Loayza (1994) and Sarte (1999), which tackle the issue of rent-seeking bureaucracies acting through excessive regulation and taxation. Amaral and Quentin (2006) tackle informality in a neoclassical competitive growth framework, where informal managers exchange physical capital for low-skilled labour, in order to cope with limited access to outside investment and financing. The consequences of a growing informal sector are thus evident for both macroeconomic and microeconomic government policy and embody a loss of effectiveness by public authorities to enforce their policies, when facing competition from a sector that acts as a substitute to government regulation, in order to be competitive in a market framework.

We finish this section by reviewing the modelling background that we will consider for our endogenous growth model. We have already discussed that we consider a small open economy model with elastic labour supply and a government sector, following Turnovsky (1999). For a background on the Turnovsky (1999) modelling proposal of endogenous growth in small open economy fiscal policy model, refer to Turnovsky (1996a, 1996b). For a detailed overview on the specific dynamics arising in endogenous growth models of open economies, following this basic framework, refer to Turnovsky (2002). For a complete overview and additional proposals on this class of models, considering both an elastic labour supply and adjustment costs, refer to Mendonça (2007).

3. The Model[6]

This economy will consist of N identical households and firms. Aggregate conditions are given by equation (A1) and households can allocate their time between labour, $\left(1-l\right)$, and leisure, l, which are set endogenously by model dynamics.

$$X_i = x_i N \ , \quad X = \sum_{i=1}^{n} X_i \ , \ i = 1,2 \tag{A1}$$

The representative agent's welfare is given by the following intertemporal isoelastic utility function, as in Turnovsky (1999). Equations (A2) and (A3) define the individual's utility and the aggregate utility for this economy, respectively:

$$U = \int_0^\infty \frac{1}{\gamma}\left(cl^\theta\right)^\gamma e^{-\rho t} dt \tag{A2}$$

$$U = \int_0^\infty \frac{1}{\gamma}\left(\left[C / N\right] l^\theta\right)^\gamma e^{-\rho t} dt \tag{A3}$$

$$\theta > 0 \ ; \ \ -\infty < \gamma < 1 \ ; \ 1 > \gamma\left(1+\theta\right) \ ; \ 1 > \gamma\theta$$

As usual C stands for aggregate private consumption, c is the household consumption, and parameters γ and θ, are related with the intertemporal elasticity of substitution and the impact of leisure on the utility of the representative agent, respectively. The constraints imposed on the parameters are necessary to ensure that the utility function is concave in c and l.

In this economy positive externalities are the result of public productive capital. Output for the individual formal firm is determined by a Samuelson type production function with non-excludable and non-rival public goods. Our proposal follows closely the well known hypotheses defined in Barro (1990) and Barro and Sala-I-Martin (1992). The formulation followed here is a variation to assume an elastic labour supply, as described in Turnovsky (1999):

[6] For reasons of simplification, we discard the use of the time subscript in the time varying variables of our model. The meaningful variables are consumption, investment, domestic capital accumulation and foreign debt accumulation. Leisure, labour allocation and government spending are given endogenously and are also subject to transitions.

$$y_1 = A G^{\beta} \left[\ell_1 \left(1 - l \right) \right]^{\phi} k_1^{1-\beta} \tag{A4}$$

$$y_1 = A \left(G / k_1 \right)^{\beta} \left[\ell_1 \left(1 - l \right) \right]^{\phi} k_1 \tag{A5}$$

$$G = gY_1 \qquad\qquad 0 < g < 1 \tag{A6}$$

$$Y_1 \left(K_1 \right) = \left(A \left[gN \right]^{\beta} \right)^{\frac{1}{1-\beta}} \left[\ell_1 \left(1 - l \right) \right]^{\frac{\phi}{1-\beta}} K_1 \tag{A7}$$

$$0 < \beta < 1,\ 0 < \phi < 1,\ \phi < \beta,\ 0 < \alpha < 1$$

Output for the individual formal firm, y_1, is given by the usual neoclassical Cobb-Douglas production function, (A4). The endogenous variable ℓ_1 defines the percentage of time devoted to working in the formal sector by our representative agent. This endogenous variable is included to ensure the normalization of the labour input to one is maintained and because it gives a straightforward option for taking an optimal control rule for labour allocation between sectors. As it is our intention to maintain the labour input normalized to one, ℓ_1 will vary between zero and one, and labour allocation in the informal sector will be just $1 - \ell_1$. As usual in economic growth formalizations, k_i denotes the individual firm capital stock and K_i the aggregate capital stock. G represents aggregate public spending and g the share the government share of formal output. Parameters A, β, and ϕ stand for the formal exogenous technology and the elasticities of government spending and labour, respectively.

In order to obtain an *AK* technology in the aggregate framework it is convenient to tie government expenditure, G, to aggregate formal output, Y_1, where g acts as an endogenously determined fraction of government expenditure relative to aggregate output. Applying aggregate conditions to the representative firm production, (A4), we obtain the aggregate formal output for this economy, as expressed in (A7). Parameter restrictions are given above and a restriction to guarantee that labour productivity is diminishing in the aggregate, $\phi < 1 - \beta$, is additionally considered.

The government sector of this economy can only observe and tax activities that occur in the formal sector. We consider, for reasons of simplification, that the public sector must always manage a balanced budget with no possibility of issuing public debt bonds. As public spending has been tied up to aggregate output, it follows that the balanced government budget constraint is given by:

$$\tau_c C + \tau_w w_1 (1 - l) \ell_1 N + \tau_k r_{k_1} K_1 + T = gY_1 \left(K_1 \right) \tag{A8}$$

Where, τ_w, represent taxes on wage income for formal sector labour allocation, τ_k, capital income taxes on revenues from entrepreneurial formal activities, τ_c, consumption taxes and finally a lump sum tax given by $\tau = T/N$. Taxes on foreign bonds, τ_b, are a possibility considered in Turnovsky (1999) that we relax in our framework. This excludes the possibility of subsidies on foreign debt accumulation, without imposing an additional parameter restriction, and has no relevant implications on the dynamic behaviour of this economy.

The representative firm in the informal sector will also have its output given by a Cobb-Douglas production function, following the same structure of (A4), where all productive factor intensities, including the elasticity of the government input, are lower than the ones considered for the formal sector production function. These hypotheses are necessary in order to obtain static equilibrium conditions for capital and labour decisions between sectors. They are also reasonable if we consider that informality arises, in order to take advantage of the absence of government regulation and taxation. Therefore, in spite of using a worst technology, they are still able to participate with success in a competitive market framework.

Following this short intuition, the production function for the representative firm in the informal sector comes:

$$y_2 = DG^{\beta - \mu}\left[\left(1-\ell_1\right)\left(1-l\right)\right]^{\xi} k_2^{\eta} \tag{A9}$$

$$Y_2 = D\left(gY_1\right)^{\beta-\mu}\left[\left(1-\ell_1\right)\left(1-l\right)\right]^{\xi} K_2^{\eta} N^{1-\eta} \tag{A10}$$

$$Y_2\left(K_1, K_2\right) = D\left(Ag\right)^{\frac{\beta-\mu}{1-\beta}} N^{\frac{\beta(\beta-\mu)+(1-\eta)(1-\beta)}{1-\beta}} \ell_1^{\frac{\phi(\beta-\mu)}{1-\beta}} \left(1-\ell_1\right)^{\xi} \left(1-l\right)^{\frac{\phi(\beta-\mu)+\xi(1-\beta)}{1-\beta}} K_1^{\beta-\mu} K_2^{\eta} \tag{A11}$$

$$0 \prec \mu \prec 1 \ , \ \beta \succ \mu \ , \ 0 \prec \xi \prec 1 \ , \ \phi \succ \xi \ , \ 0 \prec \eta \prec 1 \ , \ 1-\beta \succ \eta, A \succ D$$

Where D, represents the usual exogenous technological infrastructure, which is smaller than the one of the formal sector, A. The elasticities for the capital and labour inputs are different and restricted to be smaller than the ones in the formal sector. Labour allocation by the representative agent is given by expression $\left(1-\ell_1\right)$. Informal sector firms are still able to benefit from public goods and services, but face fiscalization of its use by the authorities, which diminishes the factor intensity for public services to be just, $\beta - \mu$. Parameter μ can therefore be used to determine the impact of fiscalization by authorities. This specification could be extended to other productive factors, where one could distinguish between factor fiscalization. In this framework, for reasons of simplification, only the dimension described above will be considered.

It is straightforward to observe that output in the informal sector will depend on output from the formal sector, due to the capacity of using public capital, though restricted by government fiscalization. As all productive factor intensities of the informal individual firm technology are lower than the ones from firms in the formal sector, investment and labour decisions in this economy will only occur if the net factor payments are equal between the two sectors. These conditions come as follows:

$$r_{k,i} = \frac{dy_i}{dk_i}, w_i = \frac{dy_i}{d(1-l)} \tag{A12}$$

$$(1-\tau_w)w_1 = w_2 \ , \ \ (1-\tau_k)r_{k_1} = r_{k_2} \tag{A13}$$

Applying the usual marginal productivity conditions described in (A12) to the equilibrium conditions described in (A13), we obtain the labour and capital market equilibrium conditions for this economy:

$$(1-\tau_w)\phi y_1 = \xi y_2 \tag{A14}$$

$$(1-\tau_k)(1-\beta)\frac{y_1}{k_1} = \eta\frac{y_2}{k_2} \tag{A15}$$

Solving (A14) for y_1 and then substituting it in (A15), we obtain static relation condition for capital between the two sectors:

$$k_2 = \Theta_1 k_1, \quad \Theta_1 = \frac{\eta(1-\tau_w)\phi}{(1-\tau_k)(1-\beta)\xi} \tag{A16}$$

Substituting (A8) in equation (A4), output of the representative firm and aggregate production for the informal sector are obtained as follows:

$$y_2(k_1) = DG^{\beta - \mu}\left[(1-\ell_1)(1-l)\right]^{\xi}\Theta_1^{\eta}k_1^{\eta} \tag{A17}$$

$$Y_2(K_1) = Dg^{\beta - \mu}\left[(1-\ell_1)(1-l)\right]^{\xi}\Theta_1^{\eta}Y_1^{\beta-\mu}K_1^{\eta}N^{1-\eta} \tag{A18}$$

Substituting Y_1 in expression (A18) and rearranging the terms, aggregate output for the informal sector is now given by the model parameters, formal sector aggregate capital and the endogenous variables governed by the model:

$$Y_2(K_1) = D\Omega\Theta_1^{\eta}K_1^{\beta + \eta - \mu} \tag{A19}$$

Where parameter Ω is equal to:

$$\Omega = (Ag)^{\frac{\beta-\mu}{1-\beta}} N^{\frac{\beta(\beta-\mu)+(1-\eta)(1-\beta)}{1-\beta}} \ell_1^{\frac{\phi(\beta-\mu)}{1-\beta}} (1-\ell_1)^{\xi} (1 - l)^{\frac{\phi(\beta-\mu)+\xi(1-\beta)}{1-\beta}} \tag{A20}$$

The following parameter restrictions must now be imposed in order to assure that capital and labour are diminishing in the aggregate:

$$\beta + \eta - \mu \prec 1 \ , \ \phi\left(\beta - \mu\right) + \xi\left(1 - \beta\right) \prec 1 - \beta \qquad \text{(A21)}$$

Applying the static market equilibrium conditions necessary for the existence of these two sectors in a competitive economy, has produced two technologies that depend only on the parameters, exogenous population employed in both sectors and capital employed in the formal sector. This style of formalization provides a clear strategy for modelling an economy, where decisions, such as capital accumulation and investment, are based solely on the formal sector variables. Informal sector capital inputs enter this economy through the static equilibrium condition given by (A16). We will consider that formal aggregate capital employed in production is always bigger than aggregate informal capital. This restriction is consistent with data and research on the informal sector size of developed economies. Recalling the market clearing condition for individual firm capital between sectors, (A16), we will just impose that $\eta\left(1 - \tau_w\right)\phi < \left(1 - \tau_k\right)\left(1 - \beta\right)\xi$, in order to guarantee that this empirical evidence is always satisfied.

Following this strategy of modelling will allow us to develop a two sector continuous time dynamic model, without having to tackle with the difficulties that arise when dealing with a two sector economy maximum problem[7]. Considering a neoclassical production function for the informal sector that depends exclusively on formal capital, will also be consistent with the existence of transitional dynamics, since total output for this economy is given separately[8]. We will deal with this subject later on, when deriving an analytical solution for the transitions dynamics of this economy, however, this subject can be described intuitively by a simple analysis of the aggregate marginal productivity of capital. Taking a partial derivative of Y_1 on K_1, we obtain:

$Y'_{K_1} = \dfrac{Y_1\left(K_1\right)}{K_1} + Y'_{2,K_1}\left(K_1\right)$, it is clear that this expression still depends on formal sector capital, which in turn depends on time.

Considering the usual endogenous growth hypotheses, where capital grows at a constant growth rate, we can obtain a simple asymptotic rule guarantying that long run output will depend only on formal sector activities:

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[7]The formalization of the two sector economy optimal control problem is reproduced in the appendix.
[8]Barro and Sala-i-Martin(1999) discuss both this strategy and the CES production function formulation, in pages 161 to 167 of their book.

$\lim_{k_1 \to \infty} Y'_{K_1} = \frac{Y_1\left(K_1\right)}{K_1}$, this result is consistent to restrictions imposed on parameters on equation (A19), which imply that the marginal productivity of aggregate informal capital will decline asymptotically, until it becomes negligible on the long run.

Assuming that capital depreciates at the constant rate equal to δ, the representative firm intertemporal capital accumulation constraint is given as usual by:

$$\dot{k}_i = i_i - \delta k_i, i = 1,2 \tag{A22}$$

In a small open economy framework, it is standard to assume that individuals and firms have full access to international capital markets and can accumulate debt (or foreign bonds) at an exogenously given world interest rate. As a result, the intertemporal budget constraint facing the representative agent in this economy will be equal to individual consumption, c, plus investment, i_i, and debt interest payments, rb, minus capital and labour incomes, $r_k k_i$ and $w_i\left(1-l\right)$, assuming that the representative agent is a net borrower[9]. To obtain this specific constraint according to the assumptions described in previous chapters, we only need to consider the additional government revenue implications, as defined in equation (A8).

$$\dot{b} = \left(1+\tau_c\right)c + i_1 + i_2 + rb + \tau - \left(1-\tau_w\right)w_1\left(1-l\right)\ell_1 - \left(1-\tau_k\right)r_{k_1}k_1 - w_2\left(1-l\right)\left(1-\ell_1\right) - r_{k_2}k_2 \tag{A23}$$

3.1. Dynamic general equilibrium conditions for labour allocation and leisure

Substituting the functional form for the representative agent utility in the optimality conditions, (A70) and (A71), for consumption and leisure, we obtain the representative agents intertemporal conditions for consumption and labour/leisure decisions:

$$c^{\gamma-1}l^{\theta\gamma} + \lambda\left(1+\tau_c\right) = 0 \tag{A24}$$

$$\theta c^{\gamma}l^{\theta\gamma-1} + \lambda\left[\left(1-\tau_w\right)w_1\ell_1 + w_2\left(1-\ell_1\right)\right] = 0 \tag{A25}$$

$$\lambda\left[-\left(1-\tau_w\right)w_1\left(1-l\right) + w_2\left(1-l\right)\right] = 0 \tag{A26}$$

Applying market clearing and aggregate conditions to expressions (A24) and (A25), substituting then the optimal control for aggregate consumption obtained from (A24), in the macroeconomic labour/leisure condition obtained from (A25), and rearranging in an intuitive form, we obtain the dynamics for labour and leisure in this economy:

[9]This type of intertemporal budget constraints leaves the possibility of analyzing agents and economies that act as net lenders also.

$$\frac{l}{\left(1-l\right)}=\frac{\theta\left(1+\tau_c\right)C}{\left(1-\tau_w\right)\phi Y_1\left(K_1\right)\ell_1+\xi Y_2\left(K_1,K_2\right)\left(1-\ell_1\right)} \tag{A27}$$

$$\lim_{t\to\infty}\frac{l}{\left(1-l\right)}=\frac{\theta\left(1+\tau_c\right)C}{\left(1-\tau_w\right)\phi Y_1\left(K_1\right)\ell_1} \tag{A28}$$

Following the same strategy for optimal labour allocation, we obtain from (A26), the expressions that determine the dynamics of labour allocation during transitions and in the long run:

$$\left(1-\tau_w\right)\phi Y_1\left(K_1\right)=\xi Y_2\left(K_1\right) \tag{A29}$$

$$\frac{\left(1-\ell_1\right)}{\ell_1}=\frac{\xi Y_2\left(K_1,K_2\right)\left(1-\ell_1\right)}{\left(1-\tau_w\right)\phi Y_1\left(K_1\right)\ell_1} \tag{A30}$$

This implies that in the long run $\lim_{t\to\infty}\left(1-\ell_1\right)=0\Rightarrow\lim_{t\to\infty}\ell_1=1$.

This section introduced the issue of informal entrepreneurial activities that are based on the existence of microeconomic assumptions, which do not hold on the aggregate framework. Recall from equation (A26), the optimality condition for labour allocation, where we obtained the same market clearing condition, (A13), that we have proposed initially for labour allocation. However equations (A29) and (A30) clearly show that at the macro level, long run equilibrium will be defined solely by formal activities and informal activities have their existence limited to transitions to the long run outcome. This concept is also observed in the behaviour of labour/leisure to equilibrium, where the long run equilibrium expression, (A28), follows the optimal condition for this class of utility functions, when $\ell_1=1$ and an optimal control problem of endogenous growth with just one state condition is considered.

3.2. Capital market equilibrium assuming no information between sectors

For the moment, we will continue to assume that our optimal control problem is given by the three state conditions described in section *1.* of the appendix. The first consequence of this assumption is obtained from the optimality conditions for investment decisions, (A73) and (A74), which state that the shadow prices of foreign and domestic capital must equalize.

$$q_1=q_2=-\lambda \tag{A31}$$

Substituting this condition in the co-state conditions, (A76) and (A77), for domestic capital accumulation, we obtain the micro equilibrium condition for domestic capital accumulation that we defined theoretically in (A13). However, as in labour allocation and leisure decisions, this condition does not hold, when the dynamic general equilibrium conditions are considered:

$$\left(1-\tau_{k}\right)\left(1-\beta\right)\frac{Y_{1}\left(K_{1}\right)}{K_{1}}=\eta\frac{Y_{2}\left(K_{1},K_{2}\right)}{K_{2}} \tag{A32}$$

Equation (A32) defines a long run equilibrium condition between formal and informal aggregate capital, which is not viable asymptotically, when we consider formal aggregate capital to grow at a constant rate and informal aggregate capital to decay asymptotically after reaching a maximum. In this case, the indifference between accumulating foreign bonds, defined in the Keynes-Ramsey consumption equation (A33), or domestic capital, would reduce to the simpler case with one formal sector of production, where informal activities would only play a role in the aggregate budget equilibrium outcome and no role in consumption decisions.

$$\dot{C}=\frac{\left(\rho-r\right)}{\left(\gamma-1\right)}C \tag{A33}$$

This assumption is contrary to empirical results on underground activities, which relate the majority share of its revenues to short and medium run consumption by their holders. This logic is related both to the motivations of agents to participate in these activities and government fiscalization limitations. Most of the participants in this type of activities accept the risk of being caught, in exchange of the possibility to expand their consumption share. On the other hand, fiscalization by authorities reinforce this incentive, due to the obvious limitations to control everyday consumption activities, where it is difficult to assert the origin of this type of revenues, without imposing excessive limitations on commercial transactions.

3.3. Capital market equilibrium assuming asymmetric information in investment decisions

To tackle the problems discussed in the previous two sections, we propose to define possible criterions to obtain an optimal control problem that is consistent with a endogenous growth model with long transitions, following the Jones and Manuelli (1990) proposal. This hypothesis is consistent to consider that the long run is a sum of short and medium run specific periods and that in all of these shorter periods, opportunities for entrepreneurial informal activities arise. As a specification consistent to the Jones and Manuelli (1990) hypothesis produces periods of long transitions to steady-state endogenous growth, it provides an ideal benchmark framework to deal with the issue of informality in a macroeconomic context. First, we know that at the micro level opportunities to engage in informal activities arise from multiple possibilities in developed economies such as excess bureaucracy, unregulated technological innovation [10], barriers to entry and our specific proposal of excess

[10] The Internet history provides a vast set of examples of informal activities that not only took advantage of existent technology but also introduced innovations, in order to obtain market shares of unregulated activities. In some cases, as the electronic music market, reached the proportions of an industrial phenomenon. In the long run, however, the majority of these operations were driven, by

taxation. Second, we know that in developed economies the formal sector activities are dominant in the long run outcomes of society, which is consistent with the outcome of an endogenous growth model driven by formal sector activities, as our proposal suggests. In order to obtain this outcome, we just need to define a different level of information available to households and firms about the relation between both productive sectors. In this section, we assume that agents have information about the linear relation between capital[11], (A16), but lack information about investment decisions between sectors.

Substituting the relation for capital between sectors, (A16), in the optimization problem described in section *1.* of the appendix, we can rewrite both the new representative agent co-state and aggregate conditions for capital accumulation easily:

$$\dot{q}_1 = \left(\rho + \Theta_2\delta\right)q_1 + \lambda\left(\left(1-\tau_k\right)r_{k_1} + \Theta_1 r_{k_2}\right) \tag{A34}$$

$$\dot{q}_1 = \left(\rho + \Theta_2\delta\right)q_1 + \lambda\left(\left(1-\tau_k\right)\left(1-\beta\right)\frac{Y_1\left(K_1\right)}{K_1} + \eta\Theta_1\frac{Y_2\left(K_1\right)}{K_1}\right) \tag{A35}$$

Where $\Theta_2 = 1 + \Theta_1 = \dfrac{\left(1-\tau_k\right)\left(1-\beta\right)\xi + \eta\left(1-\tau_w\right)\phi}{\left(1-\tau_k\right)\left(1-\beta\right)\xi}$.

In this case, we can easily observe that the remaining co-state condition for informal capital has no useful information for our optimal control problem. Therefore, we are still considering a three state optimization problem, which could be reduced to a two state problem had we decided to deal with aggregate capital accumulation instead of specific to sector capital accumulation. The results of this option, however, would produce the same analytical outcome and would not add additional hypothesis to our strategy.

To obtain the aggregate expression that is crucial to guarantee indifference in accumulation between foreign and domestic capital in this economy, we just need to consider the optimal investment condition (A73) and substitute it in the co-state conditions (A75) and (A35). Then, we can either obtain the two possible Keynes-Ramsey rules of consumption or solve the two dimensional endogenous system of co-state variables, to obtain the intertemporal financial rule for indifference in capital accumulation:

$$r = \left(1-\tau_k\right)\left(1-\beta\right)\frac{Y_1\left(K_1\right)}{K_1} + \eta\Theta_1\frac{Y_2\left(K_1\right)}{K_1} - \Theta_2\delta \tag{A36}$$

---

regulation and fiscalization by authorities, and economies of scale from formal sector firms, to adopt legal standards, face closure or reduce their activities to a residual and undetectable dimension.

[11]This assumption will imply that from now on we will consider (A17) to be the individual firm informal production technology, in substitution of (A9), and informal aggregate output to be defined by (A18) instead of (A11). This assumption is necessary in order to consider just one state condition for capital accumulation and does not alter the dynamic general equilibrium conditions considered in section *3.1.*.

$$\lim_{t\to\infty} r = \left(1-\tau_k\right)\left(1-\beta\right)\frac{Y_1\left(K_1\right)}{K_1} - \Theta_2\delta \tag{A37}$$

We take the option of not dealing with this hypothesis further and leave just this short introduction to the issue of limited information. This decision is based on the fact that the asymmetric information about investment decisions case is less tractable analytically, than the full information hypothesis of the subsequent sections. In spite of that, the strategy to solve analytically both for the long run and transitional dynamics, will follow closely that from the complete information case, although, much less intuitive. One of the main interests of this hypothesis is discussed in Mendonça (2007), where it is shown that the existence of a set of endogenous rules for the existence of an optimal fiscal policy [12], as described in Turnovsky (1999), is no longer available and public choice outcomes must arise, when a full information central planner is considered.

3.4. Capital market equilibrium assuming complete information about investment decisions

It is straightforward to obtain a linear relation for investment decisions between sectors using the linear relation for capital, already defined in (A16), and both the capital accumulation differential equations, defined by (A22). Extending the informal sector accumulation condition as a function of formal capital, following (A16), we obtain the following linear endogenous solution for investment decisions.

$$i_2 = \frac{\eta\left(1-\tau_w\right)\phi}{\left(1-\tau_k\right)\left(1-\beta\right)\xi} i_1 \tag{A38}$$

This simplifying assumption is necessary to fully internalize the information about informal activities, as a function of formal sector variables, and is consistent with the existence of a balanced growth path governing capital accumulation and no non-linear assumptions about investment decisions. Discarding capital accumulation in the informal sector and substituting the linear relation for investment in the households intertemporal open economy budget

[12] In our multiple fiscal policy framework with just a formal productive sector, optimal fiscal policy is defined by a set of endogenous linked fiscal rules that can be defined to be consistent to the neoclassical assumptions on growth maximization, no taxes on productive factors and a government size consistent with its input elasticity ($\tau_k = 0 \Rightarrow g = \beta$), guarantee that the Ramsey (1927) optimal taxation principle is achievable and endogenously given. In our specific choice of households utility, the rule that guarantees a fiscal policy that minimizes the decrement of utility, in order to minimize the economic distortion of taxation (excess burden), not taking into account the equity and redistributive aspects that may arise from fiscal policy, is given by $-\tau_w = \tau_c$. This rule defines a framework where taxes on leisure are obtained by subsidizing labour, so that both consumption and leisure, the two utility enhancing activities, are taxed uniformly resulting in a dynamic application of the Ramsey optimal taxation principle. This short description resumes the findings of Turnovsky (1999) for this class of fiscal policy endogenous growth models and builds on its main assumptions.

constraint, our optimal control problem is reduced to a two state optimal control problem with just one optimal control condition for investment decisions.

$$q_1 = -\lambda\Theta_2 \tag{A39}$$

$$\dot{q}_1 = (\rho + \delta)q_1 + \lambda\left[(1-\tau_k)r_{k_1} + \Theta_1 r_{k_2}\right] \tag{A40}$$

Applying market clearing and aggregate conditions, we obtain the dynamic general equilibrium co-state condition for formal capital:

$$\dot{q}_1 = (\rho + \delta)q_1 + \lambda\left[(1-\tau_k)(1-\beta)\frac{Y_1(K_1)}{K_1} + \eta\Theta_1\frac{Y_2(K_1)}{K_1}\right] \tag{A41}$$

Substituting the optimal control expression for investment, (A39), in (A41), and then following the same strategy for obtaining the two possible optimal Keynes-Ramsey consumption rules, we obtain the transitions and long run endogenous expressions for financial equilibrium in this economy:

$$r = \Theta_2^{-1}\left[(1-\tau_k)(1-\beta)\frac{Y_1(K_1)}{K_1} + \eta\Theta_1\frac{Y_2(K_1)}{K_1}\right] - \delta \tag{A42}$$

$$\lim_{t\to\infty} r = \Theta_2^{-1}(1-\tau_k)(1-\beta)\frac{Y_1(K_1)}{K_1} - \delta \tag{A43}$$

3.5. Long run equilibrium dynamics for the full information economy

Assuming no further hypotheses about non-convexities, the dynamic long run outcome for the complete information economy can be obtained, through solving the simple usual dynamic case of endogenous growth in a small open economy. Substituting the steady state condition, defined by (A42), in the differential equation for consumption, (A33), we obtain the intertemporal Keynes-Ramey consumption rule:

$$\dot{C} = \left(\frac{\rho + \delta - \Theta_2^{-1}\left[(1-\tau_k)(1-\beta)\frac{Y_1(K_1)}{K_1} + \eta\Theta_1\frac{Y_2(K_1)}{K_1}\right]}{\gamma - 1}\right)C \tag{A44}$$

The intertemporal aggregate budget constraint for the decentralized full information economy comes:

$$\begin{aligned}\dot{B} = {} & (1+\tau_c)C + I_1\Theta_2 + rB + T - \\ & -\left[(1-\tau_k)(1-\beta) + \phi(1-\tau_w)\ell_1\right]Y_1(K_1) - \left[\xi(1-\ell_1) + \eta\Theta_1\right]Y_2(K_1)\end{aligned} \tag{A45}$$

Substituting again the exogenous international interest rate, by the steady state rule of indifference in accumulation, (A42), investment by the capital accumulation equation and rearranging in the usual form, the net wealth differential equation for this economy comes:

$$W = \Theta_2 K_1 - B \Rightarrow \dot{W} = \Theta_2 \dot{K}_1 - \dot{B} \tag{A46}$$

$$\begin{aligned}\dot{W} = W\Theta_2^{-1}\left[\frac{Y_1(K_1)}{K_1}\left[(1-\tau_k)(1-\beta)+\phi(1-\tau_w)\ell_1\omega_{dom}\Theta_2\right]-\Theta_2\delta\right]+ \\ +\left[\frac{Y_2(K_1)}{K_1}\left(\eta\Theta_1+\xi(1-\ell_1)\omega_{dom}\Theta_2\right)\right]W\Theta_2^{-1}-(1+\tau_c)C-T\end{aligned} \tag{A47}$$

Where $\omega_{dom} = K_1 / W$.

We have considered the parameter ω_{dom} related to aggregate labour incomes, in order to simplify our system. This parameter has no transitions in the long run, when the growth rates of net wealth and formal capital equalize. We will use it to define the long run endogenous equilibrium and discuss the issue of transitions in the following sections.

We can now describe the long run endogenous equilibrium for this economy by applying the standard asymptotic assumptions, about labour allocation, production and leisure, to the system defined by (A44) and (A47). The long run dynamical system is given by:

$$\dot{C}_{l/r} = \left(\frac{\rho+\delta-\Theta_2^{-1}(1-\tau_k)(1-\beta)\dfrac{Y_1(K_1)}{K_1}}{\gamma-1}\right)C_{l/r} \tag{A48}$$

$$\dot{W}_{l/r} = \left[\frac{Y_1(K_1)}{K_1}\left[\Theta_2^{-1}(1-\tau_k)(1-\beta)+\phi(1-\tau_w)\omega_{dom}\right]-\delta\right]W_{l/r}-(1+\tau_c)C_{l/r}-T \tag{A49}$$

Solving for a common growth rate and assuming that lump-sum taxation decays asymptotically to zero in the long run, the endogenous equilibrium expressions are given by:

$$\bar{\Psi}_{l/r} = \frac{\rho+\delta-\Theta_2^{-1}(1-\tau_k)(1-\beta)\dfrac{Y_1(K_1)}{K_1}}{\gamma-1} \tag{A50}$$

$$\frac{\bar{\tilde{c}}_{l/r}}{\bar{\tilde{w}}_{l/r}} = \frac{\dfrac{Y_1(K_1)}{K_1}\left[\Theta_2^{-1}(1-\tau_k)(1-\beta)\gamma+\phi(1-\tau_w)\omega_{dom}(\gamma-1)\right]-\delta\gamma-\rho}{(\gamma-1)(1+\tau_c)} \tag{A51}$$

Where we took trends using the following scaling rule, $X_t = \tilde{x}_t e^{\Psi_x t} \Rightarrow \dot{X}_t = \dot{\tilde{x}}_t e^{\Psi_x t} + \Psi_x \tilde{x}_t e^{\Psi_x t}$.

Linearising the system around equilibrium, we can now describe the long run dynamics of this economy using the varational system:

$$\begin{bmatrix}\dot{\tilde{c}}\\ \dot{\tilde{w}}\end{bmatrix}=\begin{bmatrix}\left.\frac{d\dot{\tilde{c}}}{d\tilde{c}}\right|_{\bar{\Psi}} & \left.\frac{d\dot{\tilde{c}}}{d\tilde{w}}\right|_{\bar{\Psi}}\\ \left.\frac{d\dot{\tilde{w}}}{d\tilde{c}}\right|_{\bar{\Psi}} & \left.\frac{d\dot{\tilde{w}}}{d\tilde{w}}\right|_{\bar{\Psi}}\end{bmatrix}\begin{bmatrix}\tilde{c}-\bar{\tilde{c}}\\ \tilde{w}-\bar{\tilde{w}}\end{bmatrix}, \text{ and } J=\begin{bmatrix}\left.\frac{d\dot{\tilde{c}}}{d\tilde{c}}\right|_{\bar{\Psi}} & \left.\frac{d\dot{\tilde{c}}}{d\tilde{w}}\right|_{\bar{\Psi}}\\ \left.\frac{d\dot{\tilde{w}}}{d\tilde{c}}\right|_{\bar{\Psi}} & \left.\frac{d\dot{\tilde{w}}}{d\tilde{w}}\right|_{\bar{\Psi}}\end{bmatrix}$$

$$J=\begin{bmatrix}0 & 0\\ -\left(1+\tau_c\right) & \dfrac{\frac{Y_1\left(K_1\right)}{K_1}\left[\Theta_2^{-1}\left(1-\tau_k\right)\left(1-\beta\right)\gamma+\phi\left(1-\tau_w\right)\omega_{dom}\left(\gamma-1\right)\right]-\delta\gamma-\rho}{\left(\gamma-1\right)}\end{bmatrix}$$

Where $\det(J)=0$, $tr(J)=\dfrac{\frac{Y_1\left(K_1\right)}{K_1}\left[\Theta_2^{-1}\left(1-\tau_k\right)\left(1-\beta\right)\gamma+\phi\left(1-\tau_w\right)\omega_{dom}\left(\gamma-1\right)\right]-\delta\gamma-\rho}{\left(\gamma-1\right)}$ and the roots of the characteristic equation are given by $\lambda_1=tr(J)$ and $\lambda_2=0$.

Final restrictions for long run endogenous equilibrium in this economy come:

$$\begin{cases}\bar{\Psi}_{l/r}\succ 0\Rightarrow\rho+\delta-\Theta_2^{-1}\left(1-\tau_k\right)\left(1-\beta\right)\dfrac{Y_1\left(K_1\right)}{K_1}\prec 0\\ \bar{\tilde{c}}_{l/r},\bar{\tilde{w}}_{l/r}\succ 0\Rightarrow\dfrac{Y_1\left(K_1\right)}{K_1}\left[\Theta_2^{-1}\left(1-\tau_k\right)\left(1-\beta\right)\gamma+\phi\left(1-\tau_w\right)\omega_{dom}\left(\gamma-1\right)\right]-\delta\gamma-\rho\prec 0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\frac{\left(\rho+\delta\right)\Theta_2}{\left(1-\tau_k\right)\left(1-\beta\right)}\prec\frac{Y_1\left(K_1\right)}{K_1}\prec\frac{\rho+\delta\gamma}{\Theta_2^{-1}\left(1-\beta\right)\left(1-\tau_k\right)\gamma+\phi\left(1-\tau_w\right)\omega_{dom}\left(\gamma-1\right)}$$

The last condition for the consumption and net wealth ratio implies that this is an endogenous system that adjusts discontinuously, following the standard dynamics of linearized systems, where the *Jacobian* has a null determinant and a positive trace. However, this is only possible when we consider the asymptotic properties of this economy. To tackle the transitions for this system we need to adopt a new strategy for scaling, which is consistent with both the described asymptotic properties and long run equilibrium dynamics.

3.6. Transitional dynamics

The original proposal from Mendonça (2007) for scaling the transitions model with an informal sector, was based on the definition of a control variable and a state variable for the centralized closed economy problem, originally suggested in Jones and Manuelli (1990). This original transformation, implied considering the average product of aggregate capital, as a state like variable, and the consumption to capital ratio as a control like variable. However, in an open economy framework, an additional dimension arises, when we consider the intertemporal budget constraint. This additional state variable defines the international financial situation and introduces further transitional dynamics that can only be tackled analytically in the Jones and Manuelli (1990) fashion, if we assume some simplifying assumptions. To obtain the overall transitions of this system, we originally proposed to define the control like variable, as the consumption to net wealth ratio and the state like variable, as the average product of formal capital. In this section, we will extend our original proposal for the control like variable dynamics and assume the original proposal for a state like variable based on aggregate capital. The base transformations and respective differential equations are presented in equations (A52) and (A53):

$$Z_1 = \frac{C}{W} \Rightarrow \dot{Z}_1 = \frac{\dot{C}}{W} - Z_1 \frac{\dot{W}}{W} \tag{A52}$$

$$Z_2 = \frac{Y}{K} = \frac{Y_1\left(K_1\right) + Y_2\left(K_1\right)}{\Theta_2 K_1} \Rightarrow \dot{Z}_2 = \left[\Theta_2^{-1} \frac{d\left(Y_1\left(K_1\right) + Y_2\left(K_1\right)\right)}{dK_1} - Z_2\right] \frac{\dot{K}_1}{K_1} \tag{A53}$$

The differential equation for the state like variable Z_2 and the steady-state equilibrium expression Z_2 are given by:

$$\dot{Z}_2 = \left(\beta + \eta - \mu - 1\right)\left[Z_2 - \Theta_2^{-1} \frac{Y_1\left(K_1\right)}{K_1}\right] \frac{\dot{K}_1}{K_1} \Rightarrow \overline{Z}_2 = \Theta_2^{-1} \frac{Y_1\left(K_1\right)}{K_1} \tag{A54}$$

Recall that the growth rate expression and its transitions can be obtained as a function of Z_2:

$$\Psi = \frac{\dot{C}}{C} \Leftrightarrow \Psi\left(Z_2\right) = \frac{\rho + \delta - \left[\eta \Theta_1 Z_2 + \Theta_2^{-1} \frac{Y_1\left(K_1\right)}{K_1}\left(\left(1 - \tau_k\right)\left(1 - \beta\right) - \eta \Theta_1\right)\right]}{\gamma - 1} \tag{A55}$$

Solving for equilibrium, we obtain the long run growth rate as a function of Z_2 dynamic equilibrium defined in (A54):

$$\Psi\left(\overline{Z}_2\right) = \frac{\rho + \delta - \Theta_2^{-1}\left(1 - \tau_k\right)\left(1 - \beta\right) \frac{Y_1\left(K_1\right)}{K_1}}{\gamma - 1} \Leftrightarrow \Psi\left(\overline{Z}_2\right) = \overline{\Psi}_{l/r} \tag{A56}$$

As we have not defined any source of transitions for the growth rate of formal capital, we assume from now on that it grows at a constant rate given by:

$$\Psi_{K_1} = \Psi\left(\overline{Z}_2\right) = \overline{\Psi}_{l/r} \tag{A57}$$

This simplifying assumption is consistent with the linear assumptions about both investment decisions and capital accumulation that we have discussed in the previous sections.

Turning our attention to the control like variable defined in (A52), it is straightforward to obtain an identity that transforms this control variable into the dynamics of two specific control variables, which will able us to tackle the existent transitions for domestic capital and net wealth, already discussed in the previous section.

$$Z_1 = \frac{C}{W} \Leftrightarrow Z_1 = \frac{\frac{C}{\Theta_2 K_1}}{\frac{W}{\Theta_2 K_1}},\ Z_3 = \frac{C}{\Theta_2 K_1},\ Z_4 = \frac{W}{\Theta_2 K_1} \tag{A58}$$

Departing from (A52), assuming the term related to lump sum taxation is equal to zero, the differential equation for the control like variable comes:

$$\dot{Z}_1 = Z_1 \left\{ \frac{\rho + \delta - \left[\eta \Theta_1 Z_2 + \Theta_2^{-1} \frac{Y_1(K_1)}{K_1} \left(\left(1-\tau_k\right)\left(1-\beta\right) - \eta\Theta_1 \right) \right]}{\gamma - 1} - Z_2 \left[\eta\Theta_1 + \xi\left(1-\ell_1\right) Z_4^{-1} \right] - \right.$$
$$\left. - \Theta_2^{-1} \frac{Y_1(K_1)}{K_1} \left[\left(1-\tau_k\right)\left(1-\beta\right) + \phi\left(1-\tau_w\right)\ell_1 Z_4^{-1} - \eta\Theta_1 - \xi\left(1-\ell_1\right) Z_4^{-1} \right] + \delta + \left(1+\tau_c\right) Z_1 \right\} \tag{A59}$$

In equilibrium, excluding the obvious corner solution, we obtain the same expression as in the long run solution for the scaled capital to net wealth ratio, (A51):

$$\bar{Z}_1 = \frac{\bar{Z}_2 \left[\left(1-\tau_k\right)\left(1-\beta\right)\gamma + \phi\left(1-\tau_w\right)\left(\gamma-1\right)\ell_1 \bar{Z}_4^{-1} \right] - \delta\gamma - \rho}{\left(\gamma-1\right)\left(1+\tau_c\right)} = \frac{\bar{\tilde{c}}_{l/r}}{\bar{\tilde{w}}_{l/r}} \tag{A60}$$

From (A59) and (A60), it becomes clear that both the dynamics and equilibrium for the control like variable Z_1 are still dependent from additional transitions that are imposed by Z_4. We need to define these additional transitions in order to obtain the overall dynamics of this system.

We start by obtaining the differential equation for Z_3:

$$\dot{Z}_3 = \frac{\dot{C}}{\Theta_2 K_1} - Z_3 \frac{\dot{K}_1}{K_1} \Leftrightarrow \dot{Z}_3 = \left[\Psi\left(Z_2\right) - \Psi\left(\bar{Z}_2\right) \right] Z_3 \tag{A61}$$

The dynamics for Z_3 are given by the transitions of Z_2, as it is shown in (A61), therefore this variable is assumed to be given recursively and independently by Z_2 dynamics. A quick inspection to the dynamical system defined by (A54) and (A61), reveals that the equilibrium for the state like variable is a unique equilibrium for this system. Linearising this system in the neighbourhood of equilibrium, we obtain the eigenvalues of the Jacobian matrix, which are both real, with one equal to zero and the other equal to the trace, which is negative. By definition this is consistent with a stable degenerate focus that adjusts continuously. In our specific case, it confirms the recursive behaviour of Z_3, where the initial values, transitions and equilibrium are dependent only on the dynamics of Z_2 and the specific choice of parameters.

Finally, the dynamics for Z_4, excluding the lump-sum taxation term, are given by:

$$\dot{Z}_4 = \frac{\dot{W}}{\Theta_2 K_1} - Z_4 \frac{\dot{K}_1}{K_1} \tag{A62}$$

$$\begin{aligned}\dot{Z}_4 = Z_4 &\left\{ Z_2 \eta \Theta_1 + \Theta_2^{-1} \frac{Y_1\left(K_1\right)}{K_1}\left(\left(1-\tau_k\right)\left(1-\beta\right) - \eta\Theta_1\right) - \delta - \Psi\left(\overline{Z}_2\right) \right\} + \\ &+ Z_2 \xi \left(1-\ell_1\right) + \Theta_2^{-1} \frac{Y_1\left(K_1\right)}{K_1}\left[\phi\left(1-\tau_w\right)\ell_1 - \xi\left(1-\ell_1\right)\right] - \left(1+\tau_c\right) Z_3\end{aligned} \tag{A63}$$

In equilibrium we obtain:

$$\overline{Z}_4 = \frac{\left(\gamma - 1\right)\left[\left(1+\tau_c\right)\overline{Z}_3 - \phi\left(1-\tau_w\right)\ell_1 \overline{Z}_2\right]}{\overline{Z}_2\left(1-\tau_k\right)\left(1-\beta\right)\gamma - \rho - \delta\gamma} \tag{A64}$$

Defining a two dimensional simplified dynamical system to describe analytically the overall transitions of this economy in a clear and straightforward manner is a risky endeavour, following the extensions we have proposed in this section. Although, it is possible to define an independent system for Z_1, Z_2 and Z_4, such a solution might not capture all the dynamics involved during transitions [13], when we consider the remaining modular dynamics from the system of endogenous variables, composed by labour/leisure choices, government dimension and labour allocation decisions. However, there are some interesting features that we can add to this framework. First, we showed how it is possible to describe the control and state variable dynamic system by reducing its dimension from four to three meaningful differential equations. This approach can be performed also by using the transitions for Z_3, instead of the

[13] In section *2.* of the appendix, we define the dynamics for this three dimensional system to be saddle path dependent, when imposing a simple block reduction on the original *Jacobian* matrix, evaluated in equilibrium. However, this analysis is invalid when we assume the possible self-exciting effect of the modular endogenous system for labour allocation, leisure and government spending. It is doubtful that the Grobman-Hartman linearization theorem holds under these conditions.

transitions for Z_4, if we consider the following equality already described in (A58). Redefining $Z_1 Z_3^{-1} = Z_4^{-1}$, we can rewrite equation (A59) as:

$$\dot{Z}_1 = Z_1 \left\{ \frac{\rho + \delta - \left[\eta \Theta_1 Z_2 + \Theta_2^{-1} \frac{Y_1(K_1)}{K_1} \left((1-\tau_k)(1-\beta) - \eta \Theta_1 \right) \right]}{\gamma - 1} - Z_2 \left[\eta \Theta_1 + \xi (1 - \ell_1) Z_1 Z_3^{-1} \right] - \right. \tag{A65}$$

$$\left. -\Theta_2^{-1} \frac{Y_1(K_1)}{K_1} \left[(1-\tau_k)(1-\beta) + \phi (1-\tau_w) \ell_1 Z_1 Z_3^{-1} - \eta \Theta_1 - \xi (1-\ell_1) Z_1 Z_3^{-1} \right] + \delta + (1+\tau_c) Z_1 \right\}$$

Equilibrium for this expression will now be given by the following equation, which must be equivalent to (A60):

$$\bar{Z}_1 = \frac{\bar{Z}_2 (1-\tau_k)(1-\beta)\gamma - \delta\gamma - \rho}{(\gamma - 1)\left[(1+\tau_c) - \bar{Z}_2 \phi (1-\tau_w) \ell_1 \bar{Z}_3^{-1} \right]} \tag{A66}$$

One benefit from undertaking this option is that now we have a steady-state condition for the consumption to net wealth ratio that depends exclusively on the parameters, while in (A60) we still have to take into account the equilibrium expression for Z_4, which still depends on Z_1. The second interesting feature about this approach is that now, we are sure the reduced two dimensional original simplified system, depends only on transitions from one of the additional transformed variables. When we relax this specific transition mechanics for both simplified systems, composed by (A54) and (A59), or (A54) and (A65), we found that in both characteristic equations the roots are of opposite sign, following the rule for a negative *Jacobian* determinant. This clearly indicates that steady-state convergence dynamic associated with the transformed system should follow a unique saddle path. This possibility is reinforced when only the recursive transitions for Z_3 are considered. However, we know that both differential equations for Z_1 are non-linear, and in the case of (A65) the transformation proposed adds further non-linearities to the original proposal. Therefore, we follow a cautious path in the presence the specific complexities we enumerated and opt to present a set of proposals for numerical simulations on convergence and comparative dynamics.

We finish this section by defining the transformed algebraic system of endogenous variables:

$$\frac{l}{(1-l)} = \frac{\theta (1+\tau_c) Z_3}{Z_2 \xi (1-\ell_1) + \frac{\Theta_2^{-1} Y_1(K_1)}{K_1} \left[\ell_1 (1-\tau_w) \phi - \xi (1-\ell_1) \right]} \tag{A67}$$

$$\left[\left(1-\tau_w\right)\phi+\xi\right]\frac{\Theta_2^{-1}Y_1\left(K_1\right)}{K_1}=\xi Z_2 \tag{A68}$$

$$g=\tau_c\frac{Z_3}{\overline{Z}_2}+\tau_w\phi\ell_1+\tau_k\left(1-\beta\right) \tag{A69}$$

Again, we relax the lump sum taxation term in (A69), following all of our previous options on this issue.

3.7. Numerical convergence analysis

In the final two sections of this paper we assess the dynamic behaviour of the three dimensions transitions system, described by equations (A54), (A59) and (A63), using numerical simulation methods, with the aim of evaluating both policy and convergence outcomes. In section *2.* of the appendix, we discussed the specific dynamics obtained by evaluating the *Jacobian* for the linearized autonomous system, as being saddle path consistent. However, we also put forward the hypothesis that both long convergence paths and feedback from the endogenous system, (A67) to (A69), would likely introduce more complex dynamics. The problem was then, how to tackle this simulation in a way that would capture the overall dynamics described in section *3.6.*? As the dynamics of the differential autonomous system are most probably saddle path dependent, we decided to follow the Mendonça (2007) proposal to tackle this problem as a boundary value problem. This decision has some downfalls, but it provides a simple framework to avoid both backward error tolerant integration methods and grid search techniques, for finding consistent initial values. Then we would use a loop to evaluate the endogenous outcomes at a given step size and the respective dynamic innovations on the autonomous system, in order to evaluate the new path to the initially computed steady-state. Endogenous transitions would then impact the differential system, which in turn would produce a feed back that restarted the process. This straightforward loop structure looked promising in theory, but faced several problems when subject to experimentation. Problems arose on both endogenous steady-state estimation and differential equations collocation. As we were using the MATLAB software, specifically the old routine for numerical steady-state estimation of non-linear systems, *fsolve*, and the *bvp4c* routine, used on simulation of boundary value problems, with the purpose of tackling the proposed loop algorithm, we decided to maintain the basic framework from Mendonça (2007), of relaxing endogenous transitions and its dynamical outcomes, while extending this method just a little further and leaving the proposal for a new simulation procedure to another opportunity. In the end of this section, we discuss further options to tackle the full dynamics of this system numerically.

With the purpose of exposing the long transitions that we have discussed in the previous sections, we decided to simulate the differential autonomous system, using the standard shooting method, by assuming a boundary defined by the combined computation of the overall equilibrium system. The endogenous variables would then be relaxed to the numerical parameters obtained from the steady-state computed values. Again, problems occurred when using the *fsolve* routine with this system. This was not a surprise due to the nonlinearities present in the endogenous system. We tackled this problem by assuming labour allocation to follow the analytical and theoretical long run outcome, where all labour is considered to be allocated on formal activities. This set of assumptions provides a simplified approach to portray the long transitions and equilibrium outcomes, but embodies an excessive simplification, which holds no useful information on the outcomes of informality. We discuss some of these outcomes for policy in the next section, by assuming there is always labour allocation in the informal sector, and leave the rest of this section to discuss the outcomes of our proposal for numerical convergence analysis[14].

Choosing a set of numerical parameters, described in table 2, which are consistent with the restrictions imposed by the model, we obtain, in table 3 and in figures 1 to 5, the steady-state values and convergence dynamics for the transformed differential system, respectively.

Parameter	ρ	γ	τ_c	ℓ_1	ϕ	τ_w	β	η
Value	0,02	0,3	0,2	1	0,5	0,2	0,25	0,15
Parameter	A	μ	N	ξ	δ	τ_k	θ	Θ_1
Value	0,11	0,1	400	0,3	0,05	0,2	0,5	0.33

Table 2- Base parameters for convergence evaluation.

Z_1^*	Z_2^*	Z_3^*	Z_4^*	$\bar{\Psi}_{l/r}^*$	l^*	g^*	$\frac{Y_1(K_1)}{K_1}^*$
0,0368	0,1426	0,068	1,846	0,0222	0,4172	0,3554	0,1901

Table 3- Computed steady-state[15].

[14]All MATLAB routines used in both this and the next section simulations can be downloaded here.
[15]The numerical results obtained are sensible to the choice of initial values. Specifically, the initial choice for Z_4, holds very different steady-state outcomes, which in turn changes the remaining steady-state values also. This is an important issue, as the Z_4 variable determines the overall relative assets of the economy and a greater equilibrium value corresponds to a bigger incentive to substitute labour for leisure. Our choice of initial value for this variable was equal to 1.05 in this simulation. These outcomes are a result of non-linearities and possible different root solutions, which end up determining very different results for very similar choices of initial values.

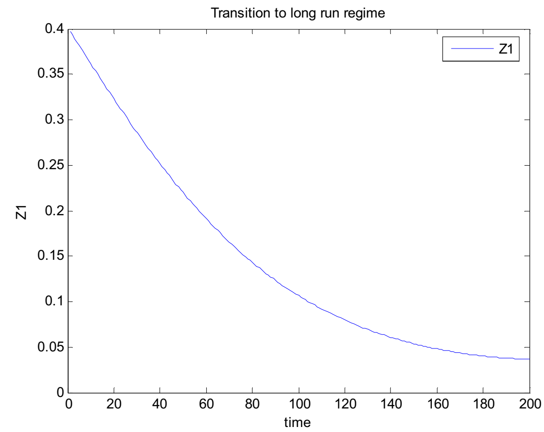

Fig. 1- Z_1 convergence transitions path.

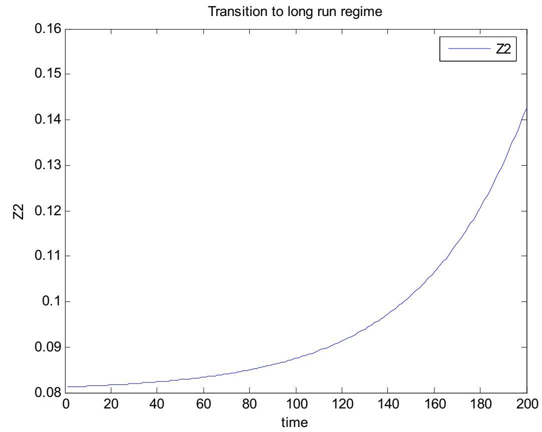

Fig. 2- Z_2 convergence transitions path.

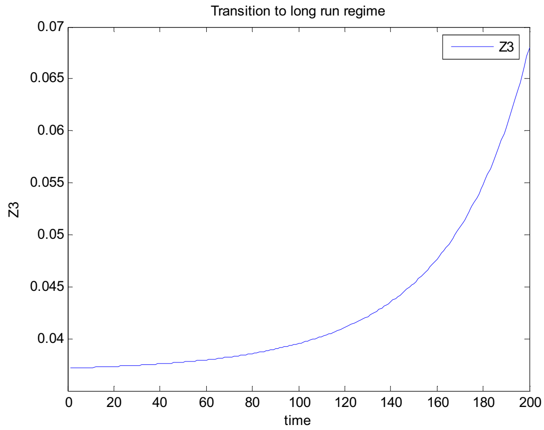

Fig. 3- Z_3 convergence transitions path.

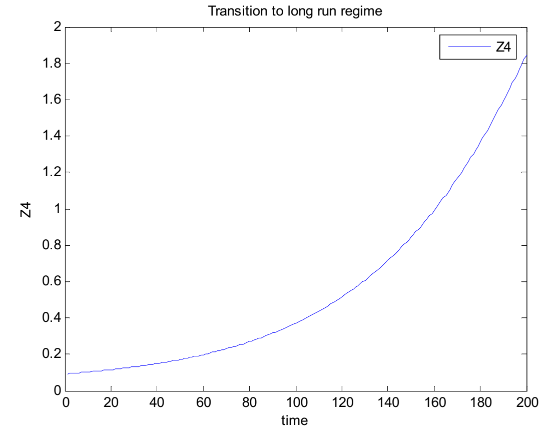

Fig. 4- Z_4 convergence transitions path.

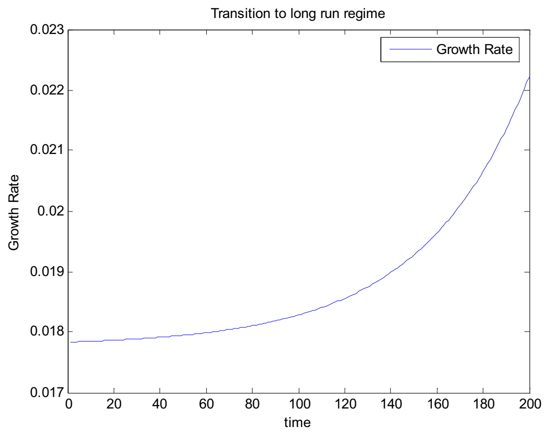

Fig. 5- Growth rate transitions during convergence.

Still, we were not able to capture the full complex dynamics of our scaled system of transitions. We have already discussed, how we were not able to implement our proposal based on the estimation of a boundary value problem assuming endogenous outcomes. Yet, there is still scope to define a new strategy that is able to tackle the overall system of transitions. Originally, this system was designed to be evaluated as a differential algebraic system. However, it is not possible to obtain the required simplified expressions for the endogenous system, without imposing several restrictions that would undermine some of the system dynamical features or, at least, violate some of the theoretical parameter restrictions. Numerical simulations strategies that build on the original idea described in this section must follow initial value methods for differential equation systems, with algebraic features. In recent economic growth theory this methods might range from simple backward integration, as proposed by Bruner and Strulik (2002), or relaxation procedures as the one proposed by Trimborn, Koch and Steger (2006). At least, such methods could be used to tackle the problem of stiff transitions arising, when boundary problem simulators are used. On the other hand, further scaling and simplification of the endogenous system might be necessary, in order to capture the full dynamics of this system with such methods. A critical hypothesis in this type of scenario would be to appeal to discretization methods and transform the overall system, in a system of partial or delay differential equations. This is always an attractive hypothesis in these problematic dynamic simulation scenarios, however, such choice would always produce outcomes that are linked with initial discretization assumptions. This option should therefore be considered as a last resort.

3.8. Comparative dynamics for fiscal policy

We end this numerical overview of the autonomous transitions system by evaluating the fiscal policy outcomes for this economy. First, we start by evaluating comparative long run outcomes of taxation policy. For this purpose, we now assume that labour allocation in the informal sector is fixed at 15%, with the aim of evaluating the long run steady-state outcomes that result from simple tax adjustments on the proposed scenario. Table 4 displays the long run outcomes for the transformed variables, when a single specific tax cut equal to 1% is considered:

	ΔZ_1^*	ΔZ_2^*	ΔZ_3^*	ΔZ_4^*	$\Delta \bar{\Psi}_{l/r}^*$	Δl^*	Δg^*	$\Delta \frac{Y_1(K_1)}{K_1}^*$
$\Delta\tau_k < 0$	>0	<0	<0	<0	>0	<0	<0	<0
$\Delta\tau_c < 0$	>0	<0	>0	>0	<0	<0	<0	<0
$\Delta\tau_w < 0$	>0	<0	>0	<0	<0	>0	<0	<0

Table 4- Fiscal policy long run outcomes.

These results come in line with our expectations, as only the capital taxes, which are directly related to our growth engine, produce growth enhancing outcomes. The remaining policies all render a long run growth diminishing effect, by either enhancing leisure, in the case of labour tax cuts, or diminishing investment in public services and infrastructure, by imposing restrictions on the government balanced budget constraint. Overall results show a negligible effect on long run outcomes from fiscal policy, except for the limited effect of the capital taxation policy. Moreover, all policies have negative effects on the overall marginal productivity of this economy, although, in the cases of capital and consumption taxation, this is due to the balanced budget effect exclusively. Other interesting feature of this simple policy evaluation, is the worsening of the long run financial macroeconomic position, portrayed by the consumption to net wealth ratio variable Z_1. Although, the causes vary among each specific policy, this is a clear indication that simple tax cut policies tend to enhance consumption incentives, at the expense of long run wealth accumulation in this economy. This outcome may only be overall *Pareto* efficient in the capital tax case, where there is an observed increase in the long run growth rate. Still, we cannot evaluate the full effect of the informal sector because, for the reasons already stated, we have to maintain labour allocation fixed. We know that informal activities take some part on the reduced impact of overall individual tax policy and on the direction of the specific outcomes, from our analytical framework. However, as we cannot observe the shifts in labour allocation between formal and informal activities, the full outcomes of policy are only a specific result related to the base parameter chosen for labour allocation. To tackle this problem, we propose to perform a grid search on possible outcomes, to evaluate the unique long run labour allocation outcome that minimizes a measure of steady-state variance, for each specific policy. To choose this threshold value for labour allocation, we grid search a set of possible values for labour allocation and choose the one that minimizes the overall absolute mean square deviation between steady-state outcomes. The threshold values for formal labour allocation for single tax cuts equal to 5%, on capital, consumption and labour are, respectively, 0,84, 0,84 and 0,83. These values determine the long run outcomes for formal labour allocation that reduce the impact of policy decisions. It also minimizes the probability of a policy performing positively or negatively, due to the existence of the substitution effect between formal and informal activities, arising from government taxation of formal activities. Of course, we know that an increase on labour allocation in the formal sector improves the long run outcome of this economy, so the results obtained suggest that even if the overall outcome of fiscal policy does not incentive formal labour allocation, there is still room for minimizing risks, when opting on simple tax reforms, rather than more complex ones, as we will show next.

We tackle this issue in this next experimentation, by assuming a complex fiscal reform policy, instead of the simple tax reforms discussed. We know there is still scope to produce long run *Pareto* efficient outcomes, when opting for a more complex reform, but the policy risks should be increased by the existence of informal activities opportunities that take advantage of the direct and indirect costs of taxation. Table 3 summarizes the long run outcomes of a fiscal reform that reduces taxes on capital to 15% and tries to minimize the reduction of the long run government budget outcome, by increasing taxes on consumption to 22% and taxes on labour to 23%:

	Z_1^*	Z_2^*	Z_3^*	Z_4^*	$\bar{\Psi}_{l/r}^*$	l^*	g^*	$\frac{Y_1(K_1)}{K_1}^*$
τ_k = 0,2, τ_c =0,2, τ_w = 0,2	0,0348	0,117	0,0634	1,8209	0,0003	0,4887	0,3433	0,1560
τ_k = 0,15, τ_c =0,22, τ_w = 0,23	0,0329	0,1192	0,0565	1,7194	0,0086	0,4678	0,315	0,1557

Table 5- Computed steady-states outcomes for fiscal reform policy.

As we expected, our radical policy proposal geared towards spurring growth, managed to improve, both the overall financial position of the economy in the long run, and sustain a government budget share above 30% of total output. Nonetheless, the incentives to increase the labour supply did not manage to overcome the overall negative effect of government spending reduction on marginal productivity. Overall, the fiscal reform has proved to be a *Pareto* improvement comparatively to the base scenario. The threshold for formal labour allocation to minimize policy outcomes is estimated to be equal to 0,85, which is exactly equal to the base parameter chosen for this experiment. This result comes in line with our expectations of the higher risks involved in choosing more complex policies, but also confirms that there is still scope for improvements, based on the limited results obtained from fiscal policy reform in this environment.

We finish this section by reproducing the simulation of the dynamic paths for the autonomous system variables, using again the shooting method from our previous section. However, we are now able to decrease the convergence problems, we discussed in the previous section, by assuming that Z_2 always adjusts continuously, which is in accordance to the linearization analysis performed on section *2.* of the appendix. We discard continuous adjustment for the remaining variable with a dynamic transition associated to a real negative eigenvector, Z_4, because overall transitions will be extremely long, before converging to the computed long run steady-state outcome. This set of assumptions allows for smother dynamic transitions to long run outcomes, however, it enhances the long, and still rigid, convergence dynamics that we discussed in previous sections. This last result concludes the set of hypothesis we have introduced in this paper, with the purpose of discussing the implications of fiscal policy on a growth environment facing an informal productive sector, by showing how government fiscal policy is not only undermined and increasingly risky, due to the existence of an informal sector, but the outcomes of that policy are subject to very long transitions, due to the persistence imposed by informal activities. This result leaves also the possibility that fiscal reform may be undermined by other exogenous shocks or different policy reforms, since a very long time frame of adjustment will surely subject these transition dynamics to other relevant innovations.

In figures 6 to 10 we reproduce the dynamic transitions for the fiscal reform proposal we discussed:

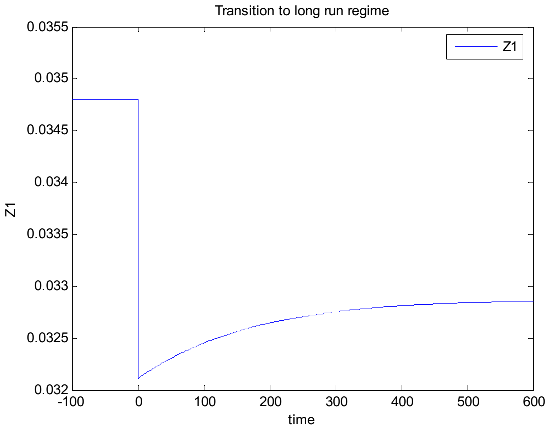

Fig. 6- Z_1 dynamic transition path.

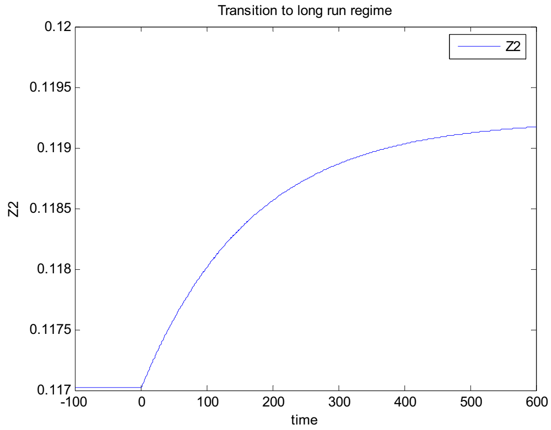

Fig. 7- Z_2 dynamic transition path.

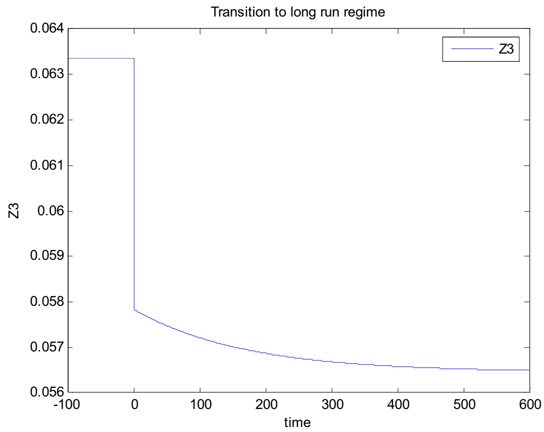

Fig. 8- Z_3 dynamic transition path.

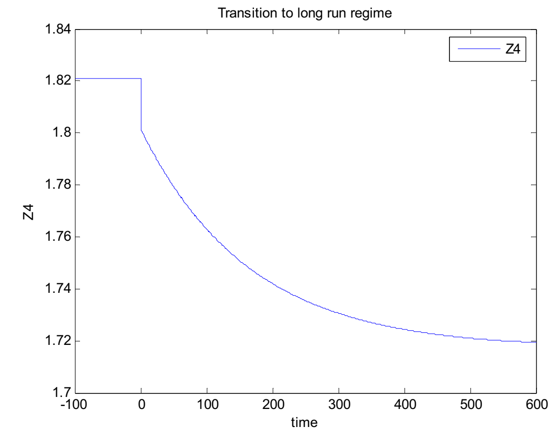

Fig. 9- Z_4 dynamic transition path.

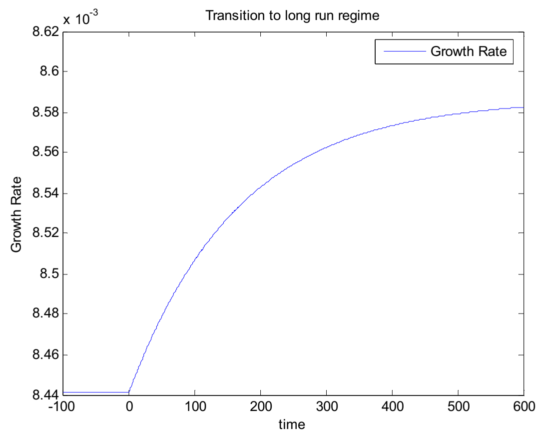

Fig. 10- Growth rate dynamic transitions.

4. Conclusions

Main stream culture has always somewhat glamorized informal activities in industrialized economies, by producing both art products, such as movies, television series, fictional literature, music lyrics, and promoting life style philosophies that portray informal activities, as an opportunity to pursue on the edge careers, as opposed to the limited opportunities for adventure, arising from being the average individual with a routine occupation. This cultural outcome is both a reflex of recent history and a part of the existent problem of increasing informal activities in developed economies, which has been identified by various social sciences researchers. However, culture is ultimately a product of society and both politics and science may claim an important share on this outcome. Politics, undermined informality issues by extending public control of the economy, based on both ideological and scientific paradigms and promoting an "under the carpet" policy to avoid dealing with all existent dimensions of this phenomenon. This led to increasing government spending and bureaucracy on the post-war period, which was not waged against emerging social practices that build up directly on excessive government intervention, or took advantage of limited fiscalization. The economic science, contributed enormously to this scenario of policies, while scientifically defining these issues as just sub-subjects of research, restricted to the field of development economics, with only some few exceptions defying this scenario. On the other hand, we have consider that the expansion of public investment, spending and regulation in developed societies, was both based on clear democratic choices by the public and various important scientific proposals, which contributed to an overall improvement of the quality of life during recent decades. This outcome has been in overall, very positive to industrialized economies, and recent academic research, public policy decisions and social awareness manifestations have already taken into account that the informal phenomena is also relevant for policy in developed economies. This change in tide has hit strongly the foundations of public policy and we no longer discuss only public investment, but are worried mainly with its efficiency. Institutions are no longer created, but designed to improve the fulfilment of policy and society objectives. Public bureaucracies are being substituted by modern efficient public services and regulated private activities. There is now a much more clear opinion that underground activities take advantage of public and private institutions, market holes and technological innovations, to develop various harmful activities that hinder economic and social outcomes. The stirring stories about the local business tycoon that started up by managing some sort of informal activity and later prospered to be a successful entrepreneur, are being scaled to their dimension, relatively to the business volume that is governed by shadow activities, which in its majority is based on the exploration of individuals and resources, with no legal, ethical and socio-economic tolerable criteria. This research proposal comes in line with these views and tries to tackle both causes and consequences of informality, in a long run growth open developed economy context. Although, we have expressed a clear critique to past policy and economics paradigms about informality, we also take note that with limited resources and information, both policy and science must adjust their priorities according to the available instruments, following contemporary goals that respond to society aspirations and concerns. So our proposal is not to ask, where did we fail in the past, but what can we do to improve the situation in the future?

We finish up by pointing out what was left to say about our proposal and where we believe it can be improved in the future. First, there is a wide scope of options for improvement of multiple fiscal policy models, both by introducing other engines of growth, such as human capital, ideas and innovation, product variety and technological change, and other sources of public spending, such as administrative costs and other productive and non-productive spending policies. Our choice for a growth engine was useful to simplify an economy with homogeneous households and firms, but it is still a limited approach to the sources of increasing returns, even if we reviewed what we considered to be physical capital, as overall measure of available capital. Unfortunately, our tangible measure has obvious limitations and cannot cope with all the different dimensions that promote growth in modern economies. Returning to our specific proposal, one could build on firm heterogeneity hypotheses to introduce sources for legal and market barriers to entry, and labour and capital income inequalities between agents. These economic dynamics, along with technological innovation and society complex evolution, are also important sources of informality, because they also promote profiteering opportunities based on shadow activities, since there are important limitations to government fiscalization. In our proposal, we suggest an optimal control framework for tackling the causes and consequences of informality in a long run growth context. As in all original two sector growth model proposals, we ended up with more questions to answer, than those we answered. We were still able to engender a full analytical proposal, based on the full information assumption about informal investment activities at the micro level. When this assumption is dismissed, we showed that a completely different outcome emerges. These modelling implications growth exponentially when we consider other simple additional features, such as investment adjustment costs and different information criteria for investment decisions. So there is still a large range of opportunities to improve our basic framework and tackle it analytically. On the issue of numerical simulations, we believe to have exhausted the possibilities of tackling this dynamic system as a boundary value problem. We discussed thoroughly the range of problems arising from our option and described a set of proposals to renew the limited simulation experiments we developed. Therefore, contributions to promote a better understanding of extended dynamical problems such as ours, are welcomed, and should be recognized as a crucial field for future economics research.

Appendix

1. The two sector economy optimal control problem

$$\underset{C,l,\ell_1,i_1,i_2}{MAX} \quad U = \int_0^\infty U(c,l)e^{-\rho t}dt$$

subject to :

$$\begin{cases} \dot{b} = (1+\tau_c)c + i_1 + i_2 + rb + \tau - (1-\tau_w)w_1(1-l)\ell_1 - (1-\tau_k)r_{k_1}k_1 - w_2(1-l)(1-\ell_1) - r_{k_2}k_2 \\ \dot{k}_1 = i_1 - \delta k_1 \\ \dot{k}_2 = i_2 - \delta k_2 \end{cases}$$

The present value Hamiltonian for this optimization problem is:

$$H^* = U(c,l) + q_1[i_1 - \delta k_1] + q_2[i_2 - \delta k_2] +$$
$$+\lambda\left[(1+\tau_c)c + i_1 + i_2 + rb + \tau - (1-\tau_w)w_1(1-l)\ell_1 - (1-\tau_k)r_{k_1}k_1 - w_2(1-l)(1-\ell_1) - r_{k_2}k_2\right]$$

The Pontryagin maximum conditions for this optimal control problem are:

Optimality Conditions

$$U_c'(c,l) + \lambda(1+\tau_c) = 0 \quad \text{(A70)}$$

$$U_l'(c,l) + \lambda\left[(1-\tau_w)w_1\ell_1 + w_2(1-\ell_1)\right] = 0 \quad \text{(A71)}$$

$$\lambda\left[-(1-\tau_w)w_1(1-l) + w_2(1-l)\right] = 0 \quad \text{(A72)}$$

$$\lambda + q_1 = 0 \quad \text{(A73)}$$

$$\lambda + q_2 = 0 \quad \text{(A74)}$$

Admissibility Conditions

$$b_0 = b_{(0)}\ ,\ k_{1,0} = k_{1,(0)} \text{ and } k_{2,0} = k_{2,(0)}$$

Multipliers Conditions

$$\dot{\lambda} = \lambda(\rho - r) \quad \text{(A75)}$$

$$\dot{q}_1 = q_1(\rho+\delta) + \lambda(1-\tau_k)r_{k_1} \quad \text{(A76)}$$

$$\dot{q}_2 = q_2(\rho+\delta) + \lambda r_{k_2} \quad \text{(A77)}$$

State Conditions

$$\dot{b} = (1+\tau_c)c + i_1 + i_2 + rb + \tau - (1-\tau_w)w_1(1-l)\ell_1 - (1-\tau_k)r_{k_1}k_1 - w_2(1-l)(1-\ell_1) - r_{k_2}k_2 \quad \text{(A78)}$$

$$\dot{k}_1 = i_1 - \delta k_1 \quad \text{(A79)}$$

$$\dot{k}_2 = i_2 - \delta k_2 \tag{A80}$$

Transversality Conditions

$$\lim_{t\to\infty} \lambda b e^{-\rho t} = 0 \tag{A81}$$

$$\lim_{t\to\infty} q_1 k_1 e^{-\rho t} = 0 \tag{A82}$$

$$\lim_{t\to\infty} q_2 k_2 e^{-\rho t} = 0 \tag{A83}$$

2. Linearised dynamics for the three dimensions system of transitions

The purpose of this section, is to provide a full description of the saddle path dynamics for the three dimensional dynamical system composed by the transitions variables $\dot{Z}_1$, $\dot{Z}_2$ and $\dot{Z}_4$, and given by equations (A59), (A54) and (A65), respectively. Substituting the term $Z_3 = Z_4 Z_1$ in equation (A65), the *Jacobian* matrix of this autonomous system evaluated in equilibrium[16] is given by:

$$J = \begin{pmatrix} \bar{Z}_1\left(1+\tau_c\right) \Rightarrow \left[J_{1,1} > 0\right] & \bar{Z}_1\left(\dfrac{-\eta\Theta_1\gamma}{\gamma-1}\right) \Rightarrow \left[J_{1,2} > 0\right] & -\bar{Z}_1\bar{Z}_2\phi\left(1-\tau_w\right)\ell_1\bar{Z}_4^{-2} \Rightarrow \left[J_{1,3} < 0\right] \\ 0 & \Psi\left(\bar{Z}_2\right)\left(\beta+\eta-\mu-1\right) \Rightarrow \left[J_{2,2} < 0\right] & 0 \\ -\bar{Z}_4\left(1+\tau_c\right) \Rightarrow \left[J_{3,1} < 0\right] & \bar{Z}_4\eta\Theta_1 \Rightarrow \left[J_{3,2} > 0\right] & -\bar{Z}_2\phi\left(1-\tau_w\right)\ell_1\bar{Z}_4^{-1} \Rightarrow \left[J_{3,3} < 0\right] \end{pmatrix}_{Z_i=\bar{Z}_i, i=1,2,3}$$

A quick inspection to the *Jacobian* renders that its determinant depends only on the diagonal terms of the matrix and is positive. The characteristic equation for this *Jacobian* is given by the following equality, where λ represents the eigenvalues associated with each specific root:

$$\left[\Psi\left(\bar{Z}_2\right)\left(\beta+\eta-\mu-1\right)-\lambda\right]\left[\left(\bar{Z}_1\left(1+\tau_c\right)-\lambda\right)\left(-\bar{Z}_2\phi\left(1-\tau_w\right)\ell_1\bar{Z}_4^{-1}-\lambda\right)-\bar{Z}_1\bar{Z}_2\phi\left(1-\tau_w\right)\ell_1\bar{Z}_4^{-1}\left(1+\tau_c\right)\right]=0$$

From the three possible solutions of the characteristic equation above, it is straightforward to obtain the eigenvalue associated with the dynamics of $\dot{Z}_2$, which is given by $J_{2,2}$. We shall define this specific root as λ_2. This root is real and negative, confirming the independent and convergent behaviour of the differential equation given by (A54). The remaining roots

[16] We extend the definition of equilibrium to the steady state outcome of the endogenous system defined by (A67), (A68) and (A69). This simplification is needed to make sure the three dimensional system is autonomous from the endogenous mechanics. Specifically, we consider that $\ell_1 = 1$, which helps simplifying the $J_{1,2}$ and $J_{3,2}$ terms. This assumption has no effect on the determinant and trace of the *Jacobian* and consequently does not influence the characteristic equation and subsequent conclusions.

associated with the dynamics of (A59) and (A65) are given by the following quadratic form solution:

$$\lambda_{1,3} = \frac{\overline{Z}_1\left(1+\tau_c\right) - \overline{Z}_2\phi\left(1-\tau_w\right)\ell_1\overline{Z}_4^{-1}}{2} \pm \frac{\sqrt{\left(\overline{Z}_1\left(1+\tau_c\right)\right)^2 + \left(\overline{Z}_2\phi\left(1-\tau_w\right)\ell_1\overline{Z}_4^{-1}\right)^2 + 6\overline{Z}_1\overline{Z}_2\phi\left(1-\tau_w\right)\ell_1\overline{Z}_4^{-1}\left(1+\tau_c\right)}}{2}$$

From this solution we can draw the conclusion that both remaining roots of the characteristic equation are real, as the square root expression is always positive, given the parameter restrictions imposed. However, the expression obtained is still too cumbersome to permit a full description of the dynamics associated with these eigenvalues. Nevertheless, we can build from the assumptions of independence on $\dot{Z}_2$ dynamics and reduce the block structure of the three dimensional *Jacobian*, in order to define the signs of the remaining roots. Our reduced form *Jacobian* is then given by the corner terms of the original *Jacobian* and is equivalent to the two dimensions *Jacobian* of the dynamical system defined by $\dot{Z}_1$ and $\dot{Z}_4$. The determinant and characteristic equation for this reduced form *Jacobian* are given by the following expressions:

$$-2\overline{Z}_2\phi\left(1-\tau_w\right)\ell_1\overline{Z}_4^{-1}\overline{Z}_1\left(1+\tau_c\right) < 0$$

$$\left(\overline{Z}_1\left(1+\tau_c\right)-\lambda\right)\left(-\overline{Z}_2\phi\left(1-\tau_w\right)\ell_1\overline{Z}_4^{-1}-\lambda\right) - \overline{Z}_1\overline{Z}_2\phi\left(1-\tau_w\right)\ell_1\overline{Z}_4^{-1}\left(1+\tau_c\right) = 0$$

From this two expressions associated with the reduced form *Jacobian,* we are able to determine that the two remaining eigenvalues of the original system have opposite signs, as the characteristic equation for this system has an equivalent solution given by the $\lambda_{1,3}$ expression and the negative determinant is associated with saddle path dynamics, with eigenvalues of opposite signs.

We can extend this dynamic assessment to the remaining two dimensions block structures of the original *Jacobian* that define the specific linearized dynamics for the $\dot{Z}_1$, $\dot{Z}_2$ and $\dot{Z}_2$, $\dot{Z}_4$ systems. For the first case, we have already observed, in the main text, that we are in the presence of saddle path dynamics, which is confirmed by the negative sign of the determinant from our reduced form *Jacobian*. In the second case, the reduced *Jacobian* has a negative trace and a positive determinant, with two negative real eigenvalues. Therefore, we are in the presence of a stable focus, with a convergent dynamic behaviour towards equilibrium. Given this analysis, we can justifiably assume that the dynamics of the global autonomous system follows a saddle path dynamic behaviour, on the three dimensional phase plane. Nevertheless, complex transitions might occur as a result of the feedback outcomes from the endogenous system, and also from the long transitions to equilibrium, originating from our modelling assumptions. Both these features may alter the signs and structure of the partial derivatives of the autonomous system *Jacobian* during transitions. Therefore, the use of linearization analysis on the varational system, following the Grobman-Hartman theorem, might be undermined by these features, which may invalidate this global dynamic analysis totally or, at least, when we depart from the close neighbourhood of equilibrium. On the other hand, this approach serves the purpose of shedding some light on the overall dynamics of this system and build up information and criteria for tackling the issues of long run transitions and endogenous feedbacks, with the use of numerical simulation procedures.

References

Agénor, Pierre-Richard, "Fiscal Policy and Endogenous Growth with Public Infrastructure", *Centre for Growth and Business Cycle Research, Economic Studies, University of Manchester,* September 2005, Discussion Paper Series nº059

Amaral, Pedro and Quintin, Erwan, "A Competitive Model of the Informal Sector", *Journal of Monetary Economics*, October 2006, 53, issue 7, 1541-1553

Barro, Robert and Sala-i-Martin, Xavier, "Economic Growth", *First MIT press edition*, Cambridge, Massachusetts, 1999

Barro, Robert and Sala-i-Martin, Xavier, "Public Finance in Models of Economic Growth", *The Review of Economic Studies*, October 1992, 59, nº4, 645-661

Barro, Robert, "Government Spending in a Simple Model of Endogeneous Growth", *The Journal of Political Economy*, October 1990, 98, nº5/2, S103-S125

Braun, Juan and Loyaza, Norman, "Taxation, Public Services and the Informal Sector in a Model of Endogenous Growth", *The World Bank, Policy of Research Department, Macroeconomic and Growth Division*, August 1994, Policy Research Working Paper nº 1334

Bruner, Martin and Strulik, Holger, "Solution of perfect foresight sadllepoint problems: a simple method and applications", *Journal of Economic Dynamics & Control*, 2002, nº26, 737-753

Frey, Bruno and Schneider, Friedrich, "Informal and Underground Economy", *International Encyclopedia of Social and Behavioural Science*, 2001

Jones, Larry and Manuelli, Rodolfo, "A Convex Model of Equilibrium Growth: Theory and Policy Implications", *The Journal of Political Economy*, October 1990, 98, nº5, part 1, 1008-1038

Jones, Larry, Manuelli, Rodolfo and Rossi, Peter, "Optimal Taxation in Models of Endogenous Growth", *The Journal of Political Economy*, June 1993, 101, nº3, 485-517

Lucas, Robert, "Supply Side Economics: An Analytical Review", *Oxford Economic Papers*, April 1990, 42 (new series), nº2, 293-296

Mendonça, Gui, "Endogenous Growth in a Small Open Economy: Optimal Fiscal Policy and the Informal Sector", *Master's Thesis,* October 2007, School of Economics and Management (ISEG), Technical University of Lisbon, http://www2.warwick.ac.uk/fac/cross_fac/comcom/dtcsite/people/students2008intake/demendonca/endogenous_growth_in_a_small_open_economy-_optimal_fiscal_policy_and_the_informal_sector.pdf

Ortigueira, Salvador, "Fiscal Policy in an Endogenous Growth Model with Human Capital Accumulation", *Journal of Monetary Economics*, 1998, 42, 323-355

Park, Hyun and Philippopoulos, Apostolis, "On the dynamics of growth and fiscal policy with redistributive transfers", *Journal of Public Economics*, 2003, 87, 515-538

Peñalosa, Cecilia and Turnovsky, Stephen, "Second-best optimal taxation of capital and labour in a developing economy", *Journal of Public Economics*, August 2004, 89, 1045-1074

Ramsey, Frank, "A Contribution to the Theory of Taxation", *The Economic Journal*, March 1927, 37, nº145, 47-61

Rebelo, Sérgio, "Long-Run Policy Analysis and Long-Run Growth", *The Journal of Political Economy*, June 1991, 99, nº3, 500-521

Romer, Paul, "Increasing Returns and Long-Run Growth", *The Journal of Political Economy*, October 1986, 94, nº5, 1002-1037

Saint-Paul, Gilles, "Fiscal Policy in an Endogenous Growth Model", *The Quarterly Journal of Economics*, November 1992, 107, nº4, 1243-1259

Sarte, Pierre-Daniel, "Informality and Rent-Seeking Bureaucracies in a Model of Long Run Growth", *Research Department, Federal Reserve Bank of Richmond*, September 1999, Working Paper 99-7

Schneider, Friedrich and Enste, Dominik, "Shadow Economies: Size, Causes, and Consequences", *Journal of Economic Literature*, March 2000, 38, 77-114

Schneider, Friedrich, "Shadow Economies Around The World: What Do We Really Know?", *European Journal of Political Economy*, March 2005, 598-642

Trimborn, Timo, Koch, Karl-Josef and Steger, Thomas, "Multi-Dimensional Transitional Dynamics: A Simple Numerical Procedure", CESifo Working Paper No. 1745**,** June 2006

Turnovsky, Stephen, "Fiscal Policy and Growth in a Small Open Economy with Elastic Labour Supply", *The Canadian Journal of Economics*, November 1999, 32, nº5, 1191-1214

Turnovsky, Stephen, "Fiscal policy, growth, and macroeconomic performance in a small open economy", *Journal of International Economics*, 1996a, 40, 41-66

Turnovsky, Stephen, "Optimal tax, debt, and expenditure policies in a Growing Economy", *Journal of Public Economics*, 1996b, 60, 21-44

Turnovsky, Stephen, "Knife-Edge Conditions and the Macrodynamics of Small Open Economies", *Macroeconomic Dynamics*, April 2002, 6, 307-335