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Ana Paula Martins

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EERI
Economics and Econometrics Research Institute
Avenue de Beaulieu
1160 Brussels
Belgium

Tel: +322 298 8491
Fax: +322 298 8490
www.eeri.eu

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Ana Paula Martins *

* Assistant Professor, Católica Lisbon School of Business & Economics, Universidade Católica Portuguesa, Cam. Palma de Cima, 1649-023 Lisboa, Portugal. Phone: 351 217214248. Fax: 351 217270252. Email: apm@ucp.pt

ABSTRACT

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This note focuses on two applications of time series methods. The first proposes a simple transformation of the unit root form of stationary testing to infer about the validity of smoothing by second-order running averages of a series, or of the variables in a linear model (here opposing co-integration testing).

The second one advances a simple iterative algorithm to correct for MA(1) autocorrelation of the residuals of the general linear model, not requiring the estimation of the error process parameter.

JEL Classification: C22, C12, C13.

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1. A Smoothing Test under First-Order Autoregressive Processes

In standard econometric as time series – undergraduate as graduate – textbooks [1] the autoregressive process of the first order is a generally covered topic and an important role is given to its correction under GLS approximations to the parameter estimators of the linear model, $Y_t = X_t \beta + \varepsilon_t$, t = 1,2,…,T. Co-integration [2] tests are usually forwarded with reference to the unit value of the AR(1) process parameter, $\varepsilon_t = \rho\, \varepsilon_{t-1} + v_t$ (v_t with null mean, constant variance and uncorrelated in time), and it is immediate to recognize that under the null (H_0: $\rho = 1$) the generalized differences form of the regression corresponds to a model estimated in first differences of all the variables involved (without a constant term) [3] (even if, as is well known, non-cointegration may be a symptom of more serious model specification problems, namely spurious regression).

It is noteworthy, however, that the null H_0: $\rho = -1$ may also be important. On the one hand, negative values for the first autoregressive coefficient also occur – in these cases, high negative, rather than positive, autocorrelation is more likely to endanger the stationarity assumption of the error term. On the other, under the null, the transformed variables can be seen to assume equivalent format to smoothed series by simple *running averages* (or moving averages, here the smoothing procedure [4]) of order 2.

The test of H_0: $\rho = -1$ against H_1: $\rho > -1$ can be easily preformed with standard co-integration statistics by means of the use of simple modifications of the error-terms regressions: for H_0: $\rho = 1$ (analogously to unit root testing), the test-statistic is based on the t-ratio associated to the coefficient of e_{t-1}, the lagged OLS residual of the co-integration relation, in either the regressions:

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[1] Greene (2003), Gujarati (2003), Griffith *et al* (1993), Johnston and Dinardo. (1997), for example.

[2] See Dolado *et al* (2001) for a recent overview.

[3] Berenblutt-Webb test confronts also the two models – in levels and in first differences. See Gujarati (2003), p. 480-481.

[4] See Mills (1990), for example, for definitions.

$\Delta e_t = (\rho - 1) e_{t-1} + \nu_t$ $\quad H_0: \rho = 1 \quad (H_1: \rho < 1$, Co-integration) or

$\Delta e_t = \mu + (\rho - 1) e_{t-1} + \nu_t$ $\quad H_0: \rho = 1 \quad (H_1: \rho < 1$, Co-integration)

$$t = 2,3,...,T$$

where $\Delta e_t = e_t - e_{t-1}$. We can generate the following reasoning for a "smoothed regression" test of $H_0: \rho = -1$ against $H_1: \rho > -1$: if $\varepsilon_t = \rho\, \varepsilon_{t-1} + v_t$, then:

$$\varepsilon_t = -\rho\ (-\varepsilon_{t-1}) + v_t = \rho'\ (-\varepsilon_{t-1}) + v_t$$

where $\rho' = -\rho$. Subtracting $(-\varepsilon_{t-1})$ from both sides of the equation, and applying the principle to OLS residuals, we can write:

(1) $$e_t + e_{t-1} = (\rho' - 1)\ (-e_{t-1}) + v_t$$

Denote the series $e_t + e_{t-1} = \Sigma e_t$, $t = 2,3,...,T$. Then, the t-ratio associated to the (only) coefficient of the regression of Σe_t on $(-e_{t-1})$ – or the symmetric of the t-ratio of the coefficient of the regression of Σe_t on e_{t-1} – can immediately be used as in the standard co-integration test to evaluate the $H_0: \rho' = 1$ (i.e., $\rho = -1$) against $H_1: \rho' < 1$ $(\rho > -1)$ [5].

One can also confirm that the estimated coefficient of $(-e_{t-1})$ – estimating $(\rho' - 1)$ - and corresponding t-ratio in regression (1) are numerically identical to the ones obtained if one replaces the original series e_t by another, e'_t, the values of which are those of e_t, say, for odd observations, and the symmetric of e_t for even ones in such a way that:

$$e'_t = -(-1)^t\, e_t \quad , \quad t = 1,2,...,T$$

(or vice-versa: $e'_t = (-1)^t\, e_t$, $t = 1,2,...,T$) - i.e., denoting by e' the vector with e'_t, and by e the vector containing the original residuals:

[5] Yet, that common co-integration tables such as those produced by Engle and Yoo (1987) are identically applicable is not proven here.

$$
e' = \begin{bmatrix} e_1' \\ e_2' \\ e_3' \\ e_4' \\ \ldots \end{bmatrix} = J\,e = \begin{bmatrix} 1 & 0 & 0 & 0 \;\ldots & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \;\ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \;\ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \;\ldots & 0 & 0 & 0 \\ & & & \ldots & & & \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \ldots \end{bmatrix}
$$

- and considers the coefficient of e'_{t-1} from the standard co-integration regression of the errors, i.e., from

$$
\Delta e'_t = (\rho' - 1)\, e'_{t-1} + v_t \qquad H_0: \rho' = 1 \quad (H_1: \rho' < 1)
$$

where $\Delta e'_t = e'_t - e'_{t-1}$. The last procedure – the inspection of the series e'_t - could be seen as stemming from Fuller's (1996) proof of Corollary 10.1.1.1., p. 554, with regard to unit root inspection. The equivalence of the two methods will not hold if we include a constant in the error-term regressions.

Finally, we note that for an AR(1) process:

$$
\varepsilon_t = \rho\, \varepsilon_{t-1} + v_t = \rho^2\, \varepsilon_{t-2} + v_t + \rho\, v_{t-1}
$$

and co-integration arrangement yields an equation of the form $(\varepsilon_t - \varepsilon_{t-2}) = (\rho^2 - 1)\, \varepsilon_{t-2} + v_t + \rho\, v_{t-1}$. $(\rho^2 - 1)$ can, thus, be inferred consistently from the regression of $(\varepsilon_t - \varepsilon_{t-2})$ on ε_{t-2}, eventually more efficiently from a procedure entailing an MA(1) correction of the error term (as the one described below); tests of H_0: $\rho^2 = 1$, accommodate both non-cointegrated as smoothing cases, the latter applicable when negative sample first-order autocorrelation of the e_t's is encountered.

The procedure(s) may easily encompass error correction mechanisms. They are also extendeable to the stationarity inspection of univariate time series, with also reference to Dickey-Fuller (1979) statistics [6]. Dickey and Fuller explicitly recognize a mirror effect on the relevant distribution for purposes of evaluating a null H_0: $\rho = -1$ with the test statistic they derived; instead, we stress the artifact(s) that allows us to obtain the "mirror" statistic that goes with the original (that is, for testing H_0: $\rho = 1$) distribution tails. If the null H_0: $\rho = -1$ is not rejected, the input series should be previously smoothed by running

[6] See Bierens (2001) for a recent survey.

averages of the second order before applying, for example, an ARMA representation. The application of the procedure to the running average series of second order would justify averaging of averages: a fourth order running average.

2. First Order Moving Average Correction

Moving average – MA(q) - processes are also de-emphasised in most textbooks in serial correlation treatment. The fact can be justified by the non-existence of a straight-forward correction procedure [7] (as generalized differences are for AR processes), generating interpretable transformed variables without requiring matrix manipulation or visualization, on the one hand, and on the other, the fact that an MA error term has, in fact, indistinguishable status in linear regression relative to the MA part of ARIMA processes in univariate time series appraisal – MA inference can be done with OLS residuals inputted to ARIMA procedures. This justifies, for instance, the non-existence of similar routines to Prais-Winsten or Cochrane-Orcutt methods [8], that correct for AR(1) serial correlation to improve estimation properties of linear model estimators, for MA processes. Yet, the correction can be accommodated quite easily, and I illustrate with an MA(1) process.

Let $\varepsilon_t = v_t + \lambda\, v_{t-1}$, where v_t stands for the usual uncorrelated, null mean and constant variance (denoted by σ_v^2) noise, invertibility requiring $|\lambda| < 1$. As is well known, and denoting by ε the (Tx1) vector of residuals ordered from 1 to T, the covariance matrix is of the form $E[\varepsilon\varepsilon'] = V = \sigma^2\,\Omega$:

$$V = \sigma_v^2 \begin{bmatrix} 1+\lambda^2 & \lambda & 0 & 0 \ldots & 0 & 0 & 0 \\ \lambda & 1+\lambda^2 & \lambda & 0 \ldots & 0 & 0 & 0 \\ 0 & \lambda & 1+\lambda^2 & \lambda \ldots & 0 & 0 & 0 \\ & & & \ldots & & & \\ 0 & 0 & 0 & 0 \ldots & \lambda & 1+\lambda^2 & \lambda \\ 0 & 0 & 0 & 0 \ldots & 0 & \lambda & 1+\lambda^2 \end{bmatrix}$$

It is straight-forward to see that V can also be written as:

---

[7] See an overview of previously proposed estimation procedures of MA(1) processes in Choudhury, Chaudhury and Power (1987), for example.

[8] As there are in packages such as TSP – see Hall and Cummins (1997) and (1998).

$$V = \sigma_v^2 (1 + \lambda^2) \begin{bmatrix} 1 & \rho_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \rho_1 & 1 & \rho_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & \rho_1 & 1 & \rho_1 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & \rho_1 & 1 & \rho_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \rho_1 & 1 \end{bmatrix}$$

where ρ_j represents the j-th order autocorrelation of ε_t, i.e., $\rho_j = \frac{E[\varepsilon_t \varepsilon_{t-j}]}{E[\varepsilon_t^2]}$. For the MA(1) process, $\rho_1 = \frac{\lambda}{1+\lambda^2}$ and $\rho_j = 0$ for $j > 1$. Then estimating ρ_1 by the sample first order autocorrelation of the estimated residuals, e_t, call it r_1, and

$$\hat{\Omega} = \begin{bmatrix} 1 & r_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ r_1 & 1 & r_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & r_1 & 1 & r_1 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & r_1 & 1 & r_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & r_1 & 1 \end{bmatrix}$$

applying standard feasible GLS, i.e., obtaining: $\hat{\beta}_{GLS} = (X' \hat{\Omega}^{-1} X)^{-1} X' \hat{\Omega}^{-1} Y$, provides adequate correction. The procedure can be iterated as in the Cochrane-Orcutt method for AR(1) residuals, with stopping rule now associated to the absolute value of the difference between two consecutive estimates of r_1 (each obtained from the errors $e_t = Y_t - X_t \hat{\beta}_{GLS}$, with $\hat{\beta}_{GLS}$ arising from the corresponding iterations).

After convergence, inference can be made using the covariance matrix $\hat{Cov}(\hat{\beta}_{MQG}) = \hat{\sigma}^2 (X' \hat{\Omega}^{-1} X)^{-1}$, with $\hat{\sigma}^2$ having been calculated with the last iteration's error terms as $\sum_{t=1}^{T} \frac{e_t^2}{T-k}$ - the residuals are homoscedastic -, with k representing the number of parameters in β.

As a final remark, we note that inference about λ was unnecessary – forecasting was not the issue here. Yet, for the underlying inferred autocorrelation to arise from an

MA(1) process with real λ and v_t's, $|\rho_1| < 0.5$ [9] – which also insures invertibility – and the method should eventually also stop if a value for r_1 out of that range is obtained.

Obviously, the procedure is immediately generalizeable to MA processes of higher order [10].

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[9] See, for example, Fuller (1996), Theorem 2.6.3., p. 66.

[10] For which much more complex methods are known in the literature - see King (2001).

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