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EERI Research Paper Series No 21/2016

ISSN: 2031-4892

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ABSTRACT

Estimation of Possibly Non-Stationary First-Order Auto-Regressive Processes

This note inspected a grid search algorithm to estimate the AR(1) process, based on the joint estimation of the canonical AR(1) equation along with its reverse form. The method relies on the GLS principle, accounting for the covariance error structure of the special estimable system. Nevertheless, it stands as potentially improving to rely on across-equation-restricted system estimation with free covariance structure.

The algorithm was (computationally) implemented and applied to inference of the AR(1) parameter of simulated – some stationary, others non-stationary - series.

Additionally, it was argued - and illustrated by simulation - that non-stationary AR(1) processes appear to be consistently estimable by OLS. Also, it was suggested that the parameter of a stationary AR(1) process is estimable by OLS from the AR(2) representation of its non-stationary "first-integrated" series; or from the joint estimate of the canonical and reverse form of the AR(1) process by OLS.

Importance of further study of differenced, D(p) – stationary after being integrated p times - processes was concluded.

JEL Classification: C22, C13 (C12), C63.

Keywords: Nonlinear Estimation; Grid Search Methods; AR(1) Processes; Integrated Series; Differenced Processes; "Factored" AR(1) Processes; Unit Roots.

Estimation of Possibly Non-Stationary First-Order Auto-Regressive Processes

1. Introduction

The first-order auto-regressive process – AR(1) - has a wide range applications in time series analysis, namely for forecasting purposes [1]. As is well-known, stationarity of the original series is required [2] for appropriate estimation, implying that the process parameter - ρ of $Y_t = \rho Y_{t-1} + v_t$ (v_t with null mean, constant variance and uncorrelated in time) – should be smaller than 1 in absolute value.

An alternative, that we could think suggested in the literature [3], is to apply the estimation procedures to an inverted form, i.e., estimating with the available algorithms under the reversed equation: $Y_{t-1} = \frac{1}{\rho} Y_t - \frac{1}{\rho} v_t$, with $-1 < \frac{1}{\rho} < 1$, applying only if $| \rho | > 1$; however, as Y_t contains v_t, the right hand-side variable is correlated with the error term – $Cov(Y_t , v'_t) = -\frac{1}{\rho} \sigma_v^2$, where v'_t represents the new error term and σ_v^2 stands for the variance of the original v_t - and minimizing the sum of square error terms of this form – the squared distances between Y_{t-1} and $\frac{1}{\rho} Y_t$ – could lead to biased estimates. The problem of direct inference of an *a priori* unrestricted parameter would remain unresolved. One of the purposes of this note was to suggest an algorithm for unique estimation of the parameter of such a process under the two forms simultaneously.

The method – and the double representation – is useful in itself, and may also allow for direct inspection of unit roots, i.e., testing of the H_0: $\rho = 1$ null. The estimation procedure, advanced below, involves a duplication of the sample size – hence, providing an extension of degrees of freedom.

In common practice, stationarity is usually sought through differencing of Y_t, say, by admitting an AR process for the differenced series. A second remark made in this note is

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[1] It is a primary reference in Time Series topics of Econometrics textbooks. See Greene (2003), Griffiths, Hill and Judge. (1993), Gujarati (2003), Johnston and Dinardo (1997), King (2001), Mills (1990), for example.

[2] Or rather, imposed or assumed.

[3] See, for example, Gonçalves e Lopes (2003).

that the parameter of non-stationary AR(1) – a | ρ | > 1 - is in fact consistently estimated by OLS – apparently, both in its primal or dual representation. On the other hand, "integrating" a stationary AR(1) series originates another that is ARI(1,1); a standard ARI(1,1) process has an equivalent AR(2) non-stationary representation – from which OLS may generate consistent estimates of the AR(1) parameter. Working with (the) integrated series may prove useful under other research [4] – namely in the case where it is the "integrated" series that is stationary itself, what we can call D(p) [5], differenced of order p, processes.

The analysis proceeds as follows: section 2 highlights the "integrated" form of a AR(1) process. Section 3 forwards the double-form system(s) of the AR(1) series; the grid-search procedure for its estimation is outlined, and, in 4, inference for the variance of the estimator. We exemplify by proceeding with simulations for AR(1) processes generated by ρ = +;_ 5 [6], +;_ 1 [7], +;_ 0.5, +;_ 0.2, and 0 in section 5; section 6 inspects the corresponding "factored" AR(1) processes. Estimation of AR(1), ARI(1,1) and ARD(1,1) processes is briefly contrasted in section 7.

2. Integrated Form of an AR(1) Process.

Let X_t follow an ARI(1,1) process with parameter *a*, i.e.

$$\Delta X_t = a \ \Delta X_{t-1} + v_t \qquad -1 < a < 1 \quad ; \qquad t = 2,3,\ldots,T$$

where $\Delta X_t = X_t - X_{t-1}$. Replace in the expression $(X_t - X_{t-1}) = a\,(X_{t-1} - X_{t-2}) + v_t$ and solving in order to X_t:

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[4] In fact, - see Dolado, Gonzalo, and Marmol (2001), p. 640 and references therein - OLS estimators of cointegrated series of order CI(1,1) are not only consistent but also super-consistent. Hence, other time series applications could benefit from the use of previously integrated processes.

[5] As opposed to I(p), integrated of order p, that should be differenced.

[6] Even if these processes are non-stationary, we will maintain the jargon "ARI(1,1)" to classify a series that once differenced generate this AR(1) process of origin. As well as "ARD(1,1)" if it is the integrated version that has such AR(1) source.

[7] See footnote 6. If the parameter is 1, one is really under an I(1) process. We caution (again) the reader for such generalization of the AR(1) qualification followed in the text, suggested to harmonize the exposition.

(1) $$X_t = (a + 1) X_{t-1} - a X_{t-2} + v_t$$

we obtain an AR(2) non-stationary representation of the ARI(1,1) process, but that, for stationarity of the differenced series, imposes special – range as well as linear combination equivalence - restrictions of the underlying parameters.

Based on this, let Y_t stand for a ARI(0,1) = AR(1) process. If that is the case, possible inference on the parameter can be accomplished by creating the corresponding "integrated" series, X_t, performing the last regression, and infer the parameter from the estimate of the coefficient of X_{t-2}.

Additionally, one concludes that testing whether the coefficient estimates of a generic AR(2) process sum 1 provides an answer to whether the first-differenced series follows an AR(1) path – the test uses the identical test statistic (generates the same restricted model under the null, the ARI(1,1) process) as the augmented-Dickey-Fuller test.

It is easy to produce integrated series – as that of differenced series - by the use of matrix manipulation. Let Y denote a T by 1 vector of time-series observations. Then, the differenced series, as is known, is generated by:

$$\mathrm{D\,Y} = \mathrm{D}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \ldots \\ Y_{T-1} \\ Y_T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \ldots \\ Y_{T-1} \\ Y_T \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 - Y_1 \\ Y_3 - Y_2 \\ \ldots \\ Y_{T-1} - Y_{T-2} \\ Y_T - Y_{T-1} \end{bmatrix}$$

The integrated series can be generated by:

$$\mathrm{S\,Y} = \mathrm{S}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \ldots \\ Y_{T-1} \\ Y_T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 1 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 1 & 1 & 1 & \ldots & 1 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \ldots \\ Y_{T-1} \\ Y_T \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_1 + Y_2 \\ Y_1 + Y_2 + Y_3 \\ \ldots \\ Y_1 + \ldots + Y_{T-2} + Y_{T-1} \\ Y_1 + \ldots + Y_{T-1} + Y_T \end{bmatrix}$$

On can show that S D = I_T (and S D Y = D S Y = Y), where I_T denotes the identity matrix of size (T x T) – that is D^{-1} = S and S^{-1} = D.

3. Estimation

Let then ρ be unknown. Both

(2) $Y_t = \rho\, Y_{t-1} + v_t$

and

(3) $Y_{t-1} = \frac{1}{\rho}\, Y_t - \frac{1}{\rho}\, v_t, \qquad t = 2,3, \dots, T$

are generically valid. Consider the application of OLS to the second equation. As Y_t and v_t are correlated the estimate of $\frac{1}{\rho}$ will be biased: replace in the second equation, equation (2):

$$Y_{t-1} = \frac{1}{\rho}\,(\rho\, Y_{t-1} + v_t) - \frac{1}{\rho}\, v_t \quad , \qquad t = 2,3, \dots, T$$

We expect [8] that plim $\frac{\sum_{t=2}^{T} Y_t Y_{t-1}}{\sum_{t=2}^{T} Y_t^2} = \frac{\frac{1}{\rho}}{1+\frac{\sigma_v^2}{Q*}}$ where Q* = plim $\frac{\rho^2 \sum_{t=2}^{T} Y_{t-1}^2}{T}$ = ρ^2 plim $\frac{\sum_{t=2}^{T} Y_{t-1}^2}{T}$. If | ρ | < 1, plim $\frac{\sum_{t=2}^{T} Y_{t-1}^2}{T}$, approaching the variance of the stationary process Y_t, will be equal to $\frac{\sigma_v^2}{1-\rho^2}$; then, plim $\frac{\sum_{t=2}^{T} Y_t Y_{t-1}}{\sum_{t=2}^{T} Y_t^2} = \frac{\frac{1}{\rho}}{1+\frac{\sigma_v^2}{\rho^2 \frac{\sigma_v^2}{1-\rho^2}}}$ = ρ and not $\frac{1}{\rho}$. (For | ρ | > 1,

[8] See Greene (2003), p. 85.

plim $\frac{\sum_{t=2}^{T} Y_{t-1}^2}{T}$ = ∞, and plim $\frac{\sum_{t=2}^{T} Y_t Y_{t-1}}{\sum_{t=2}^{T} Y_t^2} = \frac{1}{\rho}$.) As a corollary, *under stationarity*, applying OLS to (3) does in fact yield an estimate of ρ, not $\frac{1}{\rho}$. Then, on the one hand, inspecting the equality of coefficient estimates of forms (2) and (3) in real series would answer whether a series is stationary - under an AR(1) hypothesis. On the other, under stationarity, we would expect that application of OLS to:

$$\begin{bmatrix} Y_2 \\ Y_3 \\ \ldots \\ Y_T \\ Y_1 \\ Y_2 \\ \ldots \\ Y_{T-1} \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \ldots \\ Y_{T-1} \\ Y_2 \\ Y_3 \\ \ldots \\ Y_T \end{bmatrix} \text{b} + \begin{bmatrix} v_2 \\ v_3 \\ \ldots \\ v_T \\ v'_2 \\ v'_3 \\ \ldots \\ v'_T \end{bmatrix}$$

would generate a consistent estimate of ρ, plim $\hat{b}$ = plim $\frac{2\sum_{t=2}^{T} Y_t Y_{t-1}}{\sum_{t=2}^{T} Y_t^2 + \sum_{t=2}^{T} Y_{t-1}^2}$ = ρ - Fuller’s (1996) simple symmetric estimator [9]. An alternative would suggest across-equation-restricted linear system estimation with free covariance structure – standard SUR restricted estimation. As data is “duplicated”, we would eventually gain in precision relative to common practice. Yet, non-stationarity would be unresolved – and v_t and v'_t above are necessarily correlated in a particular way, suggesting the use of GLS-like, approximations. Thus, construct the system:

[9] Fuller (1996), p. 414. He also advances an alternative *weighted* symmetric estimator with analogous formatting.

$$\begin{bmatrix} Y_2 \\ Y_3 \\ \dots \\ Y_T \\ Y_1 \\ Y_2 \\ \dots \\ Y_{T-1} \end{bmatrix} = \rho \begin{bmatrix} Y_1 \\ Y_2 \\ \dots \\ Y_{T-1} \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} + \frac{1}{\rho} \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ Y_2 \\ Y_3 \\ \dots \\ Y_T \end{bmatrix} + \begin{bmatrix} v_2 \\ v_3 \\ \dots \\ v_T \\ -\frac{1}{\rho} v_2 \\ -\frac{1}{\rho} v_3 \\ \dots \\ -\frac{1}{\rho} v_T \end{bmatrix}$$

or, in pseudo-form Y = X β + ε:

(4)
$$\begin{bmatrix} Y_2 \\ Y_3 \\ \dots \\ Y_T \\ Y_1 \\ Y_2 \\ \dots \\ Y_{T-1} \end{bmatrix} = \begin{bmatrix} Y_1 & 0 \\ Y_2 & 0 \\ \multicolumn{2}{c}{\dots} \\ Y_{T-1} & 0 \\ 0 & Y_2 \\ 0 & Y_3 \\ \multicolumn{2}{c}{\dots} \\ 0 & Y_T \end{bmatrix} \begin{bmatrix} \rho \\ \frac{1}{\rho} \end{bmatrix} + \begin{bmatrix} v_2 \\ v_3 \\ \dots \\ v_T \\ -\frac{1}{\rho} v_2 \\ -\frac{1}{\rho} v_3 \\ \dots \\ -\frac{1}{\rho} v_T \end{bmatrix}$$

Being the covariance matrix of the expanded error term, vector ε, of the type:

(5)
$$\mathrm{Cov}(\varepsilon) = \sigma_v^2 \begin{bmatrix} I_{n-1} & -\frac{1}{\rho} I_{n-1} \\ -\frac{1}{\rho} I_{n-1} & \frac{1}{\rho^2} I_{n-1} \end{bmatrix} = \sigma_v^2 \; \Omega$$

It is possible to derive a matrix P that decomposes Ω into Ω = P P'. For example:

(6) $$P = \begin{bmatrix} I_{n-1} & 0 \\ -\frac{1}{\rho} I_{n-1} & 0 \end{bmatrix}$$

. We can then look for the ρ that minimizes e'e, where:

(7) $$e = GINVP \begin{bmatrix} Y_2 - \rho Y_1 \\ Y_3 - \rho Y_2 \\ \dots \\ Y_T - \rho Y_{T-1} \\ Y_1 - \frac{1}{\rho} Y_2 \\ Y_2 - \frac{1}{\rho} Y_3 \\ \dots \\ Y_{T-1} - \frac{1}{\rho} Y_T \end{bmatrix}$$

where GINVP refers the generalized inverse of P – which is singular. Alternatively, the transpose of the matrix that decomposes the generalized inverse of **Ω** [10]. Easily, a double grid search algorithm, computing e'e for values of ρ *and* of $\frac{1}{\rho}$ between –1 and 1, can provide an approximation to the parameter - and/or to its inverse.

(Note: We used for A_g, the generalized inverse of a symmetric square matrix A, $A_g = P_r D_r^{-1} P_r'$, where D_r is the diagonal matrix containing the non-zero eigenvalues of A and P_r contains the corresponding (column) eigenvectors [11].)

4. Variance

Looking at the system in matrix form (4) above, we can recognize the procedure as an approximation to a feasible GNLS (restricted) estimator. Under generalized least squares, $Cov(\hat{\beta}_{GLS}) = \sigma_v^2 (X' \Omega^{-1} X)^{-1}$; replacing in it the implied Ω^{-1} (using the

[10] See footnote 14.

[11] See Dhrymes (1978), p. 502.

pseudo-inverse again) and an estimator for σ_v^2 – obtained, for example, from by v'v/(T-1) $= \frac{\sum_{t=2}^{T}(Y_t - \rho Y_{t-1})^2}{T-1}$ - one can deduct Var(^;ρ) as the element in the first row and column of that matrix.

5. An Example

We considered the simulation of some AR(1) processes and present in the tables below of sum of squared residuals, e'e under (7), for a decimal approximation to the parameter. Of course, the algorithm cannot rehearse a white noise, i.e., ρ = 0 – because its inverse, required for (5), is infinity.

The AR(1) processes were generated for a 0 starting value of Y, i.e., $Y_1 = 0$ [12] superimposing a series of 31 (T = 31) v_t's – the same under any process - being the latter (independent) zero mean standard normal white noises [13]. For the optimal parameter (the generating parameter…), $\frac{\sum_{t=2}^{T}(Y_t - \rho Y_{t-1})^2}{T-1}$ = 0.857640861094183.

The AR(1) series were created in EXCEL; other calculations were performed with TSP [14]. Even if not reported, we essayed also with series generated by other values of ρ: +;_10. Conclusions remained unaltered.

The OLS estimator for the canonical AR(1) equation (2) invariably yielded the correct estimator – we added in the last block of rows of the table the estimated coefficient (standard error in parenthesis), the corresponding estimated variance of the residuals and the Schwarz criteria. In those last rows, we also recorded the OLS estimators of form (3), the OLS estimator of the parameter *a* of the AR(2) representation – of (1) -, i.e., of the implied/generated "integrated" series, and the estimator of *b* of the double-OLS structure.

[12] We also tested the procedure for a randomly generated starting value.

[13] Obtained by inverting the standard normal of a randomly generated series from the uniform distribution ranging between 0 and 1.

[14] CHOL() was used for matrix factorization – see Hall and Cummins (1997) and (1998). The estimation was programmed factorizing the generalized-inverse of PP' to obtain GINVP.

Table 1.
Sum of Squared Transformed Residuals, e’e
$Y_t = \rho\, Y_{t-1} + v_t$

	ρ								
^;ρ	- 5	- 1	- 0.5	-0. 2	0	0.2	0.5	1	5
-1/0.1	1.8E+308	41406.26	3645.334	2609.437	2502.323	2671.35	3507.733	46045.2	1.8E+308
-1/0.2	1.75E+26	8255.165	805.0441	626.9821	633.289	705.554	973.4968	13699.15	1.8E+308
-1/0.3	1.8E+308	2850.171	320.0114	273.5774	290.508	336.061	482.6627	7149.69	1.8E+308
-1/0.4	1.8E+308	1206.181	164.0695	154.5151	171.6607	204.897	303.6041	4668.567	1.8E+308
-1/0.5	1.8E+308	558.5096	98.2082	101.5225	117.1662	143.346	217.4036	3433.846	1.8E+308
-1/0.6	1.8E+308	267.8629	65.84393	73.87946	87.84221	109.55	168.7845	2716.513	1.8E+308
-1/0.7	1.8E+308	129.385	48.38044	57.89868	70.32792	88.7471	138.3901	2255.96	1.8E+308
-1/0.8	1.8E+308	63.33042	38.37477	47.97167	59.06874	75.1297	117.9642	1938.888	1.8E+308
-1/0.9	1.8E+308	34.35692	32.42484	41.47055	51.42364	65.6725	103.4818	1709.072	1.8E+308
- 1	1.7977E+308	25.29155	28.81923	37.0382	46.00812	58.8211	92.78075	1535.801	1.8E+308
- 0.9	1.7977E+308	27.85892	26.45146	33.63316	41.6666	53.1978	83.82218	1387.899	1.8E+308
- 0.8	1.7977E+308	40.58796	24.91481	30.78147	37.82949	48.0889	75.50066	1247.615	1.8E+308
- 0.7	1.7977E+308	63.47866	24.20927	28.48312	34.4968	43.4944	67.81617	1114.95	1.8E+308
- 0.6	1.7977E+308	96.53104	24.33485	26.73812	31.66852	39.4143	60.76873	989.9034	1.8E+308
- 0.5	1.7977E+308	139.7451	25.29154	25.54646	29.34465	35.8486	54.35833	872.4757	1.8E+308
- 0.4	1.7977E+308	193.1208	27.07935	24.90814	27.5252	32.7973	48.58497	762.6665	1.8E+308
- 0.3	1.7977E+308	256.6582	29.69828	24.82317	26.21017	30.2603	43.44865	660.4759	1.8E+308
- 0.2	1.7977E+308	330.3572	33.14832	25.29155	25.39954	28.2378	38.94936	565.9039	1.8E+308
- 0.1	1.7977E+308	414.218	37.42947	26.31326	25.09334	26.7296	35.08713	478.9506	1.8E+308
0 *	6.62068E+40	508.240364	42.54174	27.88832667	25.2915451	25.73588	31.861928	399.615797	3.04617E+40

* In this case, $\sum_{t=2}^{31} Y_t^2$ was registered.

Table 1. (Cont.)
Sum of Squared Transformed Residuals, e'e
$Y_t = \rho Y_{t-1} + v_t$

	ρ								
$\hat{\rho}$	- 5	- 1	- 0.5	-0.2	0	0.2	0.5	1	5
0.1	1.7977E+308	612.4244	48.48512	30.01674	25.99417	25.2565	29.27377	327.8997	1.8E+308
0.2	1.7977E+308	726.7701	55.25962	32.69849	27.2012	25.2915	27.32265	263.8021	1.8E+308
0.3	1.7977E+308	851.2775	62.86524	35.93358	28.91265	25.8410	26.00858	207.3232	1.8E+308
0.4	1.7977E+308	985.9466	71.30196	39.72202	31.12852	26.9048	25.33154	158.4628	1.8E+308
0.5	1.7977E+308	1130.777	80.56981	44.06381	33.8488	28.4830	25.29154	117.2211	1.8E+308
0.6	1.7977E+308	1285.77	90.66877	48.95893	37.0735	30.5756	25.88859	83.59796	1.8E+308
0.7	1.7977E+308	1450.924	101.5988	54.40741	40.80261	33.1826	27.12268	57.59344	1.8E+308
0.8	1.7977E+308	1626.24	113.36	60.40923	45.03613	36.3039	28.9938	39.20754	1.8E+308
0.9	1.7977E+308	1811.717	125.9523	66.96439	49.77407	39.9397	31.50197	28.44024	1.8E+308
1	1.8E+308	2007.356	139.3757	74.0729	55.01642	44.0899	34.64718	25.29155	1.8E+308
1/0.9	1.8E+308	2236.651	155.2654	82.62022	61.43286	49.30447	38.889	30.7284	1.8E+308
1/0.8	1.8E+308	2540.911	176.5704	94.26505	70.32912	56.71575	45.29725	50.75122	1.8E+308
1/0.7	1.8E+308	2960.906	206.3183	110.8054	83.19692	67.70257	55.34211	98.08953	1.8E+308
1/0.6	1.8E+308	3571.304	250.1048	135.604	102.856	84.90321	71.89516	198.998	1.8E+308
1/0.5	1.8E+308	4522.639	319.3213	175.5919	135.1828	113.8831	101.1365	412.828	1.8E+308
1/0.4	1.8E+308	6161.343	440.4608	247.1018	194.1815	168.0694	158.2702	892.294	1.8E+308
1/0.3	1.8E+308	9457.053	688.5331	397.0264	320.5357	286.9572	288.8842	2114.66	1.8E+308
1/0.2	1.8E+308	18165.49	1357.827	812.1556	678.3305	631.8978	682.8289	6146.602	5.21E+24
1/0.1	1.8E+308	61226.91	4750.899	2979.784	2592.406	2524.039	2926.398	30940.11	1.8E+308
SDEV($\hat{\rho}$*)	1.87156D-20	0.37064	0.95319	4.85531	4.81372	4.79722	0.92199	0.37743	2.75918D-20
SDEVI	9.35782D-21	0.041337	0.11657	0.062461	0.017750	0.028249	0.10729	0.046671	1.37959D-20
$\hat{\rho}$	-5.00000 (.640404E-08)	-.975265 (.041175)	-.665110 (.141587)	-.334644 (.175775)	-.089295 (.185215)	.143190 (.183841)	.456278 (.165271)	.991329 (.047821)	5.00000 (.202505E-08)
SSR/30	.108610E+24	.861403	.833058	.854826	.865188	.869260	.870023	.871135	.499670E+22
SBIC	839.391	41.5223	41.0205	41.4074	41.5881	41.6585	41.6717	41.6909	793.206
$\left(\frac{1}{\rho}\right)$	-.200000 (.256162E-09)	-.974964 (.041162)	-.649694 (.138305)	-.331991 (.174381)	-.089045 (.184696)	.143100 (.183725)	.456137 (.165220)	.944976 (.045585)	.200000 (.810019E-10)
SSR/30	.434441E+22	.861137	.813750	.848049	.862765	.868710	.869753	.830402	.199868E+21
SBIC	791.108	41.5177	40.6687	41.2880	41.5460	41.6490	41.6670	40.9725	744.923
$\hat{a}$	-	- .966132 (.064380)	- .662930 (.149582)	-.332463 (.183354)	- .086018 (.192881)	.147405 (.191663)	.460048 (.174280)	.979596 (.115772)	-
SSR/28		.891058	.862719	.885263	.895883	.899977	.900876	.901846	
SBIC		43.2043	42.7195	43.1064	43.2853	43.3537	43.3686	43.3848	
$\hat{b}$	-.384615 (.120174)	-.975115 (.028863)	-.657312 (.098113)	-.333312 (.122744)	-.089169 (.129670)	.143145 (.128848)	.456207 (.115852)	.967598 (.032872)	.384615 (.120174)
SSR/59	.994396E+39	.846673	.809533	.837008	.849333	.854256	.855144	.843451	.457521E+39
SBIC	2780.54	81.6860	80.3403	81.3416	81.7801	81.9536	81.9847	81.5717	2757.25

In general, all methods provided good approximations to the estimated parameter, to the exception of the single equation (3) in stationary cases – errors-in-variable problem would produce incorrect inference –, and the inference from ^;b under non-stationary ones. As expected, the estimates of ^;ρ and of $\left(\frac{1}{\rho}\right)$ would seem to coincide in these cases.

On the other hand, the AR(2) "integrated" estimate was absent – due to singularity – for non-stationary processes of parameters strictly larger than 1 in absolute value. Even if not reported, we essayed to estimate $\left(\frac{1}{a}\right)$ applying an AR(2) to the integrated process of the reverse series: only the non-stationary case was well approximated, again with estimates of $\left(\frac{1}{a}\right)$ approaching those of ^;a reported in Table 1 for stationary series.

In contrast, minimizing e'e would seem to always "converge" and provide the correct estimate, exhibiting a unique global minimum.

For the sake of completeness, we estimated, in SDEV(^;ρ*), the standard deviation of the optimal ^;ρ using the procedure described in 4 [15]. As expected, it is lower than the reported (by standard OLS) standard deviation of the parameter estimated by form (2) – which, in turn, was generally lower than that obtained under the AR(2) form – for the extreme non-stationary cases, but it was higher for the others. (However, if we consider a generalized-inverse approach to (X' Ω_g X) and only take the highest eigenvalue, we get SDEVI, always lower than OLS).

6. Factored AR(1) Processes

It is unclear whether only approximations are on the basis of our results – and mathematically, we are just reproducing OLS estimates – but through a non-linear algorithm - of equation (2) with the grid search procedure. In fact, the sum-of-square transformed residuals remained invariant to the replacement of GINVP by $\begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$. On the one hand, for non-stationary processes, non-linear restricted least-squares estimation of ρ from the system

[15] For – 0.5, this same value was used, even if not the indicated by the algorithm.

$$\begin{bmatrix} Y_2 \\ Y_3 \\ \dots \\ Y_T \\ Y_1 \\ Y_2 \\ \dots \\ Y_{T-1} \end{bmatrix} = \begin{bmatrix} Y_1 & 0 \\ Y_2 & 0 \\ \dots & \\ Y_{T-1} & 0 \\ 0 & Y_2 \\ 0 & Y_3 \\ \dots & \\ 0 & Y_T \end{bmatrix} \begin{bmatrix} \rho \\ \frac{1}{\rho} \end{bmatrix} + \begin{bmatrix} v_2 \\ v_3 \\ \dots \\ v_T \\ v'_2 \\ v'_3 \\ \dots \\ v'_T \end{bmatrix}$$

would apparently enjoy the same properties as Fuller's (1996) simple symmetric estimator justified above for stationary processes. As before, allowing for a free covariance structure across the two equations might promote the improvement of estimates precision.

On the other, the mechanics remain interesting specially for estimation under a modified – say, "factored" - AR(1) process with the form:

(8) $$Y_t = \rho\, Y_{t-1} + \rho\, v_t$$

If (8) and not (2) applies, (3) becomes $Y_{t-1} = \frac{1}{\rho} Y_t - v_t$ and $Q = \begin{bmatrix} \rho I_{n-1} & 0 \\ -I_{n-1} & 0 \end{bmatrix}$ could be used instead of P of (6) to transform the error vector.

Under ignorance about whether (2) or (8) are correct, alternatives weighing both possibilities can be devised; hypothetically, an "unrestricted" estimated P could be a better approximation.

We completed the analysis reproducing the calculations of Table 1 for data generated by (8) from the same random shocks. Results are summarized in Table 2; Table 2.A contains the sum of square errors of the grid search simulations "inversely" factored, i.e., divided (back, to derive the total corresponding v'v) by ρ^2 (first and last columns not reported due to inappropriate approximations).

Table 2. Sum of Squared Transformed Residuals, e'e $Y_t = \rho Y_{t-1} + \rho v_t$								
	ρ							
^;ρ	- 5	- 1	- 0.5	-0. 2	0.2	0.5	1	5
-1/0.1	1.8E+308	41406.26	911.3334	104.3775	106.854	876.9333	46045.2	1.8E+308
-1/0.2	1.75E+26	8255.165	201.261	25.07928	28.22215	243.3742	13699.15	1.8E+308
-1/0.3	1.8E+308	2850.171	80.00285	10.94309	13.44244	120.6657	7149.69	1.8E+308
-1/0.4	1.8E+308	1206.181	41.01738	6.180603	8.195893	75.90102	4668.567	1.8E+308
-1/0.5	1.8E+308	558.5096	24.55205	4.0609	5.733818	54.35091	3433.846	1.8E+308
-1/0.6	1.8E+308	267.8629	16.46098	2.955178	4.378208	42.19611	2716.513	1.8E+308
-1/0.7	1.8E+308	129.385	12.09511	2.315947	3.549885	34.59752	2255.96	1.8E+308
-1/0.8	1.8E+308	63.33042	9.593692	1.918867	3.00519	29.49105	1938.888	1.8E+308
-1/0.9	1.8E+308	34.35692	8.10621	1.658822	2.626898	25.87046	1709.072	1.8E+308
- 1	1.8E+308	25.29155	7.204806	1.481528	2.352843	23.19519	1535.801	1.8E+308
- 0.9	1.8E+308	27.85892	6.612864	1.345326	2.127911	20.95555	1387.899	1.8E+308
- 0.8	1.8E+308	40.58796	6.228702	1.231259	1.923556	18.87516	1247.615	1.8E+308
- 0.7	1.8E+308	63.47866	6.052318	1.139325	1.739776	16.95404	1114.95	1.8E+308
- 0.6	1.8E+308	96.53104	6.083713	1.069525	1.576572	15.19218	989.9034	1.8E+308
- 0.5	1.8E+308	139.7451	6.322886	1.021858	1.433943	13.58958	872.4757	1.8E+308
- 0.4	1.8E+308	193.1208	6.769838	0.996326	1.31189	12.14624	762.6665	1.8E+308
- 0.3	1.8E+308	256.6582	7.42457	0.992927	1.210413	10.86216	660.4759	1.8E+308
- 0.2	1.8E+308	330.3572	8.287079	1.011662	1.129511	9.737341	565.9039	1.8E+308
- 0.1	1.8E+308	414.218	9.357368	1.052531	1.069185	8.771782	478.9506	1.8E+308
0 *	1.65517E+42	508.2404	10.63543	1.115533	1.029435	7.965482	399.6158	7.62E+41

* In this case, $\sum_{t=2}^{31} Y_t^2$ was registered.

Table 2. (Cont.)
Sum of Squared Transformed Residuals, e'e
$Y_t = \rho\, Y_{t-1} + \rho\, v_t$

	ρ							
^;ρ	- 5	- 1	- 0.5	-0.2	0.2	0.5	1	5
0.1	1.8E+308	612.4244	12.12128	1.200669	1.010261	7.318442	327.8997	1.8E+308
0.2	1.8E+308	726.7701	13.81491	1.307939	1.011662	6.830663	263.8021	1.8E+308
0.3	1.8E+308	851.2775	15.71631	1.437343	1.033639	6.502144	207.3232	1.8E+308
0.4	1.8E+308	985.9466	17.82549	1.588881	1.076191	6.332885	158.4628	1.8E+308
0.5	1.8E+308	1130.777	20.14245	1.762552	1.139319	6.322886	117.2211	1.8E+308
0.6	1.8E+308	1285.77	22.66719	1.958357	1.223023	6.472147	83.59796	1.8E+308
0.7	1.8E+308	1450.924	25.39971	2.176296	1.327303	6.780669	57.59344	1.8E+308
0.8	1.8E+308	1626.24	28.34001	2.416369	1.452158	7.248451	39.20754	1.8E+308
0.9	1.8E+308	1811.717	31.48808	2.678576	1.597589	7.875493	28.44024	1.8E+308
1	1.8E+308	2007.356	34.84394	2.962916	1.763595	8.661796	25.29155	1.8E+308
1/0.9	1.8E+308	2236.651	38.81636	3.304809	1.972179	9.722251	30.7284	1.8E+308
1/0.8	1.8E+308	2540.911	44.1426	3.770602	2.26863	11.32431	50.75122	1.8E+308
1/0.7	1.8E+308	2960.906	51.57958	4.432216	2.708103	13.83553	98.08953	1.8E+308
1/0.6	1.8E+308	3571.304	62.5262	5.424159	3.396128	17.97379	198.998	1.8E+308
1/0.5	1.8E+308	4522.639	79.83031	7.023676	4.555323	25.28413	412.828	1.8E+308
1/0.4	1.8E+308	6161.343	110.1152	9.884072	6.722775	39.56755	892.294	1.8E+308
1/0.3	1.8E+308	9457.053	172.1333	15.88106	11.47829	72.22104	2114.66	1.8E+308
1/0.2	1.8E+308	18165.49	339.4567	32.48622	25.27591	170.7072	6146.602	5.21E+24
1/0.1	1.8E+308	61226.91	1187.725	119.1913	100.9616	731.5994	30940.11	1.8E+308
^;ρ	-5.00000 (.937005E-08)	-.975265 (.041175)	-.665110 (.141587)	-.334644 (.175775)	.143190 (.183841)	.456278 (.165271)	.991329 (.047821)	5.00000 (.238171E-08)
SSR/30	.581282E+25	.861403	.208265	.034193	.034770	.217506	.871135	.172795E+24
$\left(\frac{1}{\rho}\right)$	-.200000 (.374802E-09	-.974964 (.041162)	-.649694 (.138305)	-.331991 (.174381)	.143100 (.183725)	.456137 (.165220)	.944976 (.045585)	.200000 (.952684E-10)
SSR/30	.232513E+24	.861137	.203438	.033922	.034748	.217438	.830402	.691179E+22
^;a	-	-.966132 (.064380)	-.662930 (.149582)	-.332463 (.183354)	.147405 (.191663)	.460048 (.174280)	.979596 (.115772)	-
SSR/28		.891058	.215680	.035411	.035999	.225219	.901846	
^;b	-.384615 (.120174)	-.975115 (.028863)	-.657312 (.098113)	-.333312 (.122744)	.143145 (.128848)	.456207 (.115852)	.967598 (.032872)	.384615 (.120174)
SSR/59	.248599E+41	.846673	.202383	.033480	.034170	.213786	.843451	.114380E+41

Table 2.A.

Factored Sum of Squared Transformed Residuals, $\frac{1}{\rho^2}$ e'e

$Y_t = \rho Y_{t-1} + \rho v_t$

	ρ							
^;ρ	- 5	- 1	- 0.5	-0. 2	0.2	0.5	1	5
-1/0.1		414.0626	9.113335	1.043775	1.06854	8.769333	460.452	
-1/0.2		330.2066	8.050442	1.003171	1.128886	9.734967	547.9659	
-1/0.3		256.5154	7.200256	0.984878	1.20982	10.85991	643.4722	
-1/0.4		192.989	6.56278	0.988896	1.311343	12.14416	746.9707	
-1/0.5		139.6274	6.138012	1.015225	1.433454	13.58773	858.4615	
-1/0.6		96.43063	5.925954	1.063864	1.576155	15.1906	977.9448	
-1/0.7		63.39864	5.926604	1.134814	1.739444	16.95278	1105.42	
-1/0.8		40.53147	6.139963	1.228075	1.923321	18.87427	1240.888	
-1/0.9		27.8291	6.56603	1.343646	2.127788	20.95507	1384.348	
- 1		25.29155	7.204806	1.481528	2.352843	23.19519	1535.801	
- 0.9		34.39372	8.16403	1.660897	2.627051	25.87104	1713.455	
- 0.8		63.41868	9.732347	1.923842	3.005556	29.49244	1949.398	
- 0.7		129.5483	12.35167	2.325152	3.550563	34.60009	2275.408	
- 0.6		268.1418	16.8992	2.970901	4.379365	42.20051	2749.732	
- 0.5		558.9803	25.29154	4.087433	5.735772	54.35833	3489.903	
- 0.4		1207.005	42.31149	6.227035	8.199313	75.91401	4766.666	
- 0.3		2851.758	82.49522	11.03252	13.44903	120.6907	7338.621	
- 0.2		8258.931	207.177	25.29155	28.23778	243.4335	14147.6	
- 0.1		41421.8	935.7368	105.2531	106.9185	877.1782	47895.05	
0 *	6.62068E+40	508.2404	42.54174	27.88833	25.73588	7.965482	1598.463	3.04617E+40

* In this case, $\frac{1}{\rho^2} \sum_{t=2}^{31} Y_t^2$ was registered.

Table 2.A (Cont.) Factored Sum of Squared Transformed Residuals, $\frac{1}{\rho^2}$ e’e $Y_t = \rho\, Y_{t-1} + \rho\, v_t$								
	ρ							
^;ρ	- 5	- 1	- 0.5	-0.2	0.2	0.5	1	5
0.1		61242.44	1212.128	20.0669	101.0261	731.8442	32789.96	
0.2		18169.25	345.3727	2.69849	25.29154	170.7666	6595.052	
0.3		9458.64	174.6257	5.97048	11.48487	72.24604	2303.591	
0.4		6162.167	111.4093	.930505	6.726194	39.58053	990.3926	
0.5		4523.109	80.56981	.050209	4.557277	25.29154	468.8843	
0.6		3571.583	62.96442	.439882	3.397286	17.97819	232.2166	
0.7		2961.069	51.83614	.441421	2.708781	13.8381	117.5376	
0.8		2540.999	44.28126	.775577	2.268996	11.3257	61.26178	
0.9		2236.688	38.87418	.306883	1.972332	9.722831	35.1114	
1		2007.356	34.84394	2.962916	1.763595	8.661796	25.29155	
1/0.9		1811.687	31.44125	2.676895	1.597465	7.875023	24.89001	
1/0.8		1626.183	28.25127	2.413185	1.451923	7.247561	32.48078	
1/0.7		1450.844	25.274	2.171786	1.32697	6.779408	48.06387	
1/0.6		1285.669	22.50943	1.952697	1.222606	6.470565	71.63927	
1/0.5		1130.66	19.95758	1.755919	1.138831	6.321032	103.207	
1/0.4		985.8148	17.61843	1.581452	1.075644	6.330808	142.767	
1/0.3		851.1348	15.492	1.429295	1.033046	6.499894	190.3194	
1/0.2		726.6195	13.57827	1.299449	1.011036	6.82829	245.8641	
1/0.1		612.2691	11.87725	1.191913	1.009616	7.315994	309.4011	

Finally, a second “factored” AR(1) process was inspected:

(9) $$Y_t = \rho\, Y_{t-1} + \frac{1}{\rho}\, v_t$$

We reproduce in Table 3 the same calculations as before for series generated according to equation (9); Table 3.A contains the sum of square errors of the grid search simulations “inversely” factored, i.e., multiplied (back, to derive the total corresponding v’v) by ρ^2.

Table 3.
Sum of Squared Transformed Residuals, e'e

$$Y_t = \rho\, Y_{t-1} + \frac{1}{\rho}\, v_t$$

^;ρ \ ρ	- 5	- 1	- 0.5	-0. 2	0.2	0.5	1	5
-1/0.1	1.8E+308	41406.26	14581.33	65235.91	66783.78	14030.93	46045.2	1.8E+308
-1/0.2	1.38E+24	8255.165	3220.177	15674.55	17638.84	3893.987	13699.15	1.8E+308
-1/0.3	2.94E+38	2850.171	1280.046	6839.434	8401.528	1930.651	7149.69	1.8E+308
-1/0.4	1.8E+308	1206.181	656.278	3862.876	5122.434	1214.416	4668.567	1.8E+308
-1/0.5	1.8E+308	558.5096	392.8328	2538.062	3583.636	869.6145	3433.846	1.8E+308
-1/0.6	1.8E+308	267.8629	263.3757	1846.986	2736.38	675.1378	2716.513	1.8E+308
-1/0.7	1.8E+308	129.385	193.5218	1447.467	2218.678	553.5602	2255.96	1.8E+308
-1/0.8	1.8E+308	63.33042	153.4991	1199.292	1878.244	471.8568	1938.888	1.8E+308
-1/0.9	1.8E+308	34.35692	129.6994	1036.764	1641.812	413.9274	1709.072	1.8E+308
- 1	1.8E+308	25.29155	115.2769	925.9548	1470.527	371.123	1535.801	1.8E+308
- 0.9	1.8E+308	27.85892	105.8058	840.8289	1329.945	335.2887	1387.899	1.8E+308
- 0.8	1.8E+308	40.58796	99.65923	769.5366	1202.222	302.0026	1247.615	1.8E+308
- 0.7	1.8E+308	63.47866	96.83709	712.0779	1087.36	271.2647	1114.95	1.8E+308
- 0.6	1.8E+308	96.53104	97.3394	668.4528	985.3573	243.0749	989.9034	1.8E+308
- 0.5	1.8E+308	139.7451	101.1662	638.6614	896.2144	217.4333	872.4757	1.8E+308
- 0.4	1.8E+308	193.1208	108.3174	622.7035	819.9313	194.3399	762.6665	1.8E+308
- 0.3	1.8E+308	256.6582	118.7931	620.5793	756.5081	173.7946	660.4759	1.8E+308
- 0.2	1.8E+308	330.3572	132.5933	632.2886	705.9446	155.7975	565.9039	1.8E+308
- 0.1	1.8E+308	414.218	149.7179	657.8316	668.2409	140.3485	478.9506	1.8E+308
0 *	2.65E+39	508.2404	170.167	697.2082	643.397	127.4477	399.6158	1.22E+39

* In this case, $\sum_{t=2}^{31} Y_t^2$ was registered.

Table 3. (Cont.)
Sum of Squared Transformed Residuals, e'e

$$Y_t = \rho Y_{t-1} + \frac{1}{\rho} v_t$$

	ρ							
^;ρ	- 5	- 1	- 0.5	-0.2	0.2	0.5	1	5
0.1	1.8E+308	612.4244	193.9405	750.4183	631.4129	117.0951	327.8997	1.8E+308
0.2	1.8E+308	726.7701	221.0385	817.4622	632.2886	109.2906	263.8021	1.8E+308
0.3	1.8E+308	851.2775	251.461	898.3395	646.0242	104.0343	207.3232	1.8E+308
0.4	1.8E+308	985.9466	285.2079	993.0505	672.6194	101.3262	158.4628	1.8E+308
0.5	1.8E+308	1130.777	322.2792	1101.595	712.0745	101.1662	117.2211	1.8E+308
0.6	1.8E+308	1285.77	362.6751	1223.973	764.3894	103.5544	83.59796	1.8E+308
0.7	1.8E+308	1450.924	406.3954	1360.185	829.5641	108.4907	57.59344	1.8E+308
0.8	1.8E+308	1626.24	453.4401	1510.231	907.5986	115.9752	39.20754	1.8E+308
0.9	1.8E+308	1811.717	503.8093	1674.11	998.4929	126.0079	28.44024	1.8E+308
1	1.8E+308	2007.356	557.503	1851.822	1102.247	138.5887	25.29155	1.8E+308
1/0.9	1.8E+308	2236.651	621.0617	2065.506	1232.612	155.556	30.7284	1.8E+308
1/0.8	1.8E+308	2540.911	706.2817	2356.626	1417.894	181.189	50.75122	1.8E+308
1/0.7	1.8E+308	2960.906	825.2733	2770.135	1692.564	221.3684	98.08953	1.8E+308
1/0.6	1.8E+308	3571.304	1000.419	3390.099	2122.58	287.5807	198.998	1.8E+308
1/0.5	1.8E+308	4522.639	1277.285	4389.797	2847.077	404.546	412.828	1.8E+308
1/0.4	1.8E+308	6161.343	1761.843	6177.545	4201.734	633.0807	892.294	3.05E+38
1/0.3	1.8E+308	9457.053	2754.133	9925.659	7173.929	1155.537	2114.66	1.35E+38
1/0.2	1.8E+308	18165.49	5431.307	20303.89	15797.44	2731.316	6146.602	3.78E+23
1/0.1	1.8E+308	61226.91	19003.6	74494.59	63100.98	11705.59	30940.11	1.8E+308
^;ρ	-5.00000 (.112083E-07)	-.975265 (.041175)	-.665110 (.141587)	-.334644 (.175775)	.143190 (.183841)	.456278 (.165271)	.991329 (.047821)	5.00000 (.100834E-07
SSR/30	.133076E+23	.861403	3.33223	21.3707	21.7315	3.48009	.871135	.495553E+22
$\left(\frac{1}{\rho}\right)$	-.200000 (.448330E-09)	-.974964 (.041162)	-.649694 (.138305)	-.331991 (.174381)	.143100 (.183725)	.456137 (.165220)	.944976 (.045585)	.200000 (.403337E-09
SSR/30	.532302E+21	24.9730	3.25500	21.2012	21.7177	3.47901	.830402	.198221E+21
^;a	-	-.966132 (.064380)	-.662930 (.149582)	-.332463 (.183354)	.147405 (.191663)	.460048 (.174280)	.979596 (.115772)	-
SSR/28		.891058	3.45088	22.1316	22.4994	3.60350	.901846	
^;b	-.384615 (.120174)	-.975115 (.028863)	-.657312 (.098113)	-.333312 (.122744)	.143145 (.128848)	.456207 (.115852)	.967598 (.032872)	.384615 (.120174)
SSR/59	.397759E+38	.846673	3.23813	20.9252	21.3564	3.42057	.843451	.183008E+38

Table 3.A.

Factored Sum of Squared Transformed Residuals, ρ^2 e’e

$$Y_t = \rho\, Y_{t-1} + \frac{1}{\rho}\, v_t$$

	ρ							
^;ρ	- 5	- 1	- 0.5	-0. 2	0.2	0.5	1	5
-1/0.1	1.8E+308	4140626	1458134	6523592	6678379	1403093	4604520	1.8E+308
-1/0.2	3.45E+25	206379.1	80504.41	391863.8	440971.1	97349.67	342478.7	1.8E+308
-1/0.3	1.8E+308	31668.57	14222.73	75993.71	93350.31	21451.68	79441.01	1.8E+308
-1/0.4	1.8E+308	7538.633	4101.738	24142.98	32015.21	7590.103	29178.54	1.8E+308
-1/0.5	1.8E+308	2234.039	1571.331	10152.25	14334.54	3478.458	13735.38	1.8E+308
-1/0.6	1.8E+308	744.0635	731.5992	5130.518	7601.055	1875.383	7545.871	1.8E+308
-1/0.7	1.8E+308	264.051	394.9424	2954.014	4527.915	1129.715	4604	1.8E+308
-1/0.8	1.8E+308	98.95378	239.8423	1873.893	2934.755	737.2763	3029.512	1.8E+308
-1/0.9	1.8E+308	42.41595	160.1227	1279.955	2026.928	511.0215	2109.965	1.8E+308
- 1	1.8E+308	25.29155	115.2769	925.9548	1470.527	371.123	1535.801	1.8E+308
- 0.9	1.8E+308	22.56572	85.70273	681.0714	1077.255	271.5839	1124.198	1.8E+308
- 0.8	1.8E+308	25.97629	63.78191	492.5034	769.4224	193.2817	798.4735	1.8E+308
- 0.7	1.8E+308	31.10455	47.45017	348.9182	532.8064	132.9197	546.3254	1.8E+308
- 0.6	1.8E+308	34.75117	35.04219	240.643	354.7286	87.50697	356.3652	1.8E+308
- 0.5	1.8E+308	34.93627	25.29154	159.6653	224.0536	54.35833	218.1189	1.8E+308
- 0.4	1.8E+308	30.89933	17.33079	99.63256	131.189	31.09438	122.0266	2.27E+38
- 0.3	2.11E+38	23.09924	10.69138	55.85213	68.08572	15.64151	59.44283	1.23E+38
- 0.2	9.76E+37	13.21429	5.30373	25.29154	28.23778	6.231898	22.63616	5.27E+37
- 0.1	2.54E+37	4.142179	1.497179	6.578316	6.682409	1.403485	4.789505	1.27E+37
0 *	6.62068E+40	508.2404	42.54174	27.88833	25.73588	127.4477	99.90395	3.04617E+40

* In this case, $\rho^2 \sum_{t=2}^{31} Y_t^2$ was registered.

Table 3.A (Cont.)

Factored Sum of Squared Transformed Residuals, ρ^2 e'e

$$Y_t = \rho\, Y_{t-1} + \frac{1}{\rho}\, v_t$$

	ρ							
^;ρ	- 5	- 1	- 0.5	-0.2	0.2	0.5	1	5
0.1	2.76E+37	6.124244	1.939405	7.504183	6.314129	1.170951	3.278996	1.17E+37
0.2	1.15E+38	29.07081	8.841539	32.69849	25.29155	4.371624	10.55208	4.49E+37
0.3	2.68E+38	76.61498	22.63148	80.85056	58.14217	9.363087	18.65908	9.69E+37
0.4	1.8E+308	157.7515	45.63326	158.8881	107.6191	16.21218	25.35405	1.65E+38
0.5	1.8E+308	282.6943	80.56981	275.3988	178.0186	25.29154	29.30527	2.47E+38
0.6	1.8E+308	462.8771	130.563	440.6304	275.1802	37.27957	30.09527	3.4E+38
0.7	1.8E+308	710.9527	199.1337	666.4907	406.4864	53.16045	28.22079	1.8E+308
0.8	1.8E+308	1040.793	290.2017	966.5476	580.8631	74.22414	25.09282	1.8E+308
0.9	1.8E+308	1467.491	408.0855	1356.029	808.7792	102.0664	23.03659	1.8E+308
1	1.8E+308	2007.356	557.503	1851.822	1102.247	138.5887	25.29155	1.8E+308
1/0.9	1.8E+308	2761.297	766.7429	2550.007	1521.743	192.0445	37.9363	1.8E+308
1/0.8	1.8E+308	3970.174	1103.565	3682.228	2215.459	283.1078	79.29878	1.8E+308
1/0.7	1.8E+308	6042.665	1684.231	5653.336	3454.213	451.7723	200.1827	1.8E+308
1/0.6	1.8E+308	9920.288	2778.942	9416.941	5896.056	798.8352	552.7722	1.8E+308
1/0.5	1.8E+308	18090.55	5109.14	17559.19	11388.31	1618.184	1651.312	1.8E+308
1/0.4	1.8E+308	38508.39	11011.52	38609.66	26260.84	3956.755	5576.837	1.8E+308
1/0.3	1.8E+308	105078.4	30601.47	110285.1	79710.33	12839.3	23496.22	1.8E+308
1/0.2	1.8E+308	454137.2	135782.7	507597.2	394936.1	68282.89	153665	9.45E+24
1/0.1	1.8E+308	6122691	1900360	7449459	6310099	1170559	3094011	1.8E+308

Inspection of the Tables leads us to conclude that factored forms should not have different treatment than the standard ones.

The AR(2) "integrated" estimate was, again, not obtainable by OLS – due to singularity – for non-stationary processes of parameters strictly larger than 1 in absolute value. Yet, a "stationary" parameter was adequately approximated. This would suggest the convenience of the additional inspection of the AR(2) form of integrated series in current estimation of AR(1) processes.

7. Other Series

We also inspected briefly the estimates of AR(1) and AR(2) processes with a constant, a trend, and both, for the same generated AR(1) series, their integrated form, an ARI(1,1) process, and their first-difference, an "ARD(1,1)". Results are depicted in the tables below, where we shade in each column the lowest residual variance and Schwarz criteria, along with a graphical representation of the three series for ρ = - 0.2 and + 0.2.

A priori, their "original generating form" – after differencing for I(.) processes; after "integration" for D(.) processes - would suggest the estimation of $\hat{\rho}$ by OLS depicted in Table 1; that is the case for the AR(1) process – reported in Table 4.A: the residual variance and Schwarz criteria in the corresponding estimation form of Table 1 are always lower than those of Table 4.A.

For the ARI(1,1) process, one attains a lower residual variance and Schwarz criteria – Table 4.B - in other than the OLS form of Table 1.

Non-stationary AR1 coefficients are always stably estimated, but in this case, non-singularity is found in AR(2) estimation forms.

An ARD(1,1) process – that is, one which after "first-integration" is an AR(1) process, illustrated in Table 4.C – appears often as an AR(2) one. The minimum criteria are found for the integrated series in Table 1. However, in practice, we generally disregard the integration option – even if not differencing.

D(p) – "differenced" of order p - processes may have a clear conceptual advantage for the analysis of "flow" variables that accumulate over time in a meaningful stock. That is the case of inflation; its integrated (log or similar) form may approach the evolvement of the price level itself. Also of investment series (which contribute to a stock of capital). Of interest rates, in some compound applications. In general, Y_t follows a D(1) process if its accumulated sum, $\Sigma Y_t = \Sigma Y_{t-1} + Y_t$, follows another known process (eventually, restricted to be stationary).

Finally, rejection of a unit root test applied to the "first-integrated" series of Y_t, ΣY_t, justifies the use of "first-integration". On the other hand, notice that the case for which ρ = 1 – Table 4.C, column before last - reverts the process ΣY_t to the original series; then, the AR(1) coefficient estimate (without constant nor trend) is around zero.

Table 4.A
Sum of Squared Transformed Residuals, e'e
$Y_t \sim AR(1)$ Process: $Y_t = \rho Y_{t-1} + v_t$

	ρ								
	- 5	- 1	- 0.5	-0.2	0	0.2	0.5	1	5
Original Series									
Interc.	-.328860E+11 (.616136E+11)	-.142196 (.170375)	-.156926 (.167317)	-.160490 (.170185)	-.159635 (.172115)	-.159711 (.173805)	-.170974 (.177962)	-.422445 (.263101)	-.104372E+11 (.133302E+11)
AR coef.	-5.00000 (.655777E-08)	-.975926 (.041400)	-.673492 (.142163)	-.353267 (.177215)	-.114971 (.187716)	.110628 (.187712)	.408804 (.172711)	.899334 (.073835)	5.00000 (.209166E-08)
SSR/28	.111356E+24	.870512	.836530	.858102	.869378	.873950	.872339	.826177	.506428E+22
SBIC	840.940	42.8544	42.2571	42.6390	42.8348	42.9135	42.8858	42.0703	794.582
Trend	-.288542E+10 (.318927E+10)	-.00476711 (.00878782)	-.00509391 (.00863561)	-.00529142 (.00876063)	-.00528732 (.00883605)	-.00528299 (.00889356)	-.00563893 (.00904452)	-.049516 (.023034)	-.885806E+09 (.702505E+09)
AR coef.	-5.00000 (.662717E-08)	-.975596 (.041690)	-.667219 (.143250)	-.341583 (.178103)	-.099968 (.188147)	.128675 (.187526)	.432415 (.171370)	.737939 (.126203)	5.00000 (.215209E-08)
SSR/28	.109294E+24	.882889	.852220	.873969	.884773	.889100	.888757	.774436	.489708E+22
SBIC	840.659	43.0661	42.5358	42.9138	43.0981	43.1713	43.1655	41.1001	794.078
Interc.	.851450E+11 (.139623E+12)	-.298944 (.389870)	-.345391 (.383053)	-.344112 (.389380)	-.337886 (.393151)	-.335139 (.395628)	-.346080 (.397879)	-.013118 (.400136)	.226591E+11 (.299771E+11)
Trend	-.687560E+10 (.729507E+10)	.895656E-02 (.019968)	.010746 (.019587)	.010446 (.019861)	.010122 (.020007)	.994004E-02 (.020071)	.989220E-02 (.020034)	-.048613 (.036183)	-.197590E+10 (.160654E+10)
AR coef.	-5.00000 (.679638E-08)	-.976033 (.042004)	-.679109 (.144335)	-.360876 (.180131)	-.123208 (.190956)	.102171 (.191058)	.402047 (.175625)	.739705 (.139341)	5.00000 (.223858E-08)
SSR/27	.111802E+24	.896076	.857949	.880858	.893110	.898159	.896551	.803086	.497322E+22
SBIC	842.155	44.4436	43.7914	44.1867	44.3939	44.4784	44.4515	42.8001	795.465
Interc.	-.340467E+11 (.650463E+11)	-.125884 (.176799)	-.089999 (.173241)	-.083175 (.174573)	-.088123 (.177880)	-.098312 (.181580)	-.121481 (.188649)	-.356250 (.281249)	-.108167E+11 (.140771E+11)
AR1 coef.	-5.00000 (.680676E-08)	-1.09615 (.190711)	-.547945 (.188688)	-.267804 (.186679)	-.086773 (.188154)	.101402 (.190209)	.391812 (.191982)	.839168 (.193563)	5.00000 (.217172E-08)
AR2 coef.	-	-.123412 (.190776)	.193492 (.194431)	.241069 (.191299)	.207127 (.191813)	.148296 (.193141)	.067698 (.195358)	.078897 (.189526)	-
SSR/26	.119879E+24	.886074	.833124	.837214	.861767	.885254	.900217	.864220	.544949E+22
SBIC	815.158	42.8629	41.9695	42.0405	42.4596	42.8495	43.0925	42.5008	770.339
Trend	-.288784E+10 (.331101E+10)	-.00497549 (.00887850)	-.00342441 (.00866209)	-.00315649 (.00870938)	-.00336769 (.00886090)	-.00376829 (.00902609)	-.00459921 (.00933564)	-.048542 (.024569)	-.886610E+09 (.729336E+09)
AR1 coef.	-5.00000 (.687743E-08)	-1.08595 (.190116)	-.539371 (.187430)	-.260038 (.185393)	-.078840 (.186945)	.110054 (.189083)	.402681 (.190905)	.767795 (.192040)	5.00000 (.223341E-08)
AR2 coef.	-	-.113200 (.190182)	.201717 (.193303)	.248173 (.190271)	.214035 (.190966)	.155381 (.192526)	.075185 (.195507)	-.026309 (.190686)	-
SSR/26	.117699E+24	.892570	.836742	.840278	.865095	.889274	.906116	.797774	.527351E+22
SBIC	814.892	42.9688	42.0323	42.0934	42.5155	42.9152	43.1872	41.3408	769.863
Interc.	.962531E+11 (.155342E+12)	-.201105 (.427599)	-.154573 (.415950)	-.142428 (.417901)	-.147474 (.424795)	-.161434 (.431728)	-.195295 (.437399)	.201784 (.459654)	.256435E+11 (.334059E+11)
Trend	-.737546E+10 (.798015E+10)	.00415005 (.021395)	.00355824 (.020753)	.00326111 (.020811)	.00326294 (.021120)	.00346516 (.021412)	.00405346 (.021575)	-.063257 (.041792)	-.211364E+10 (.175941E+10)
AR1 coef.	-5.00000 (.706851E-08)	-1.09985 (.195278)	-.551272 (.193288)	-.270868 (.191284)	-.089823 (.192802)	.098171 (.194900)	.388075 (.196654)	.764811 (.195211)	5.00000 (.233207E-08)
AR2 coef.	-	-.127150 (.195361)	.191349 (.198558)	.239605 (.195216)	.205994 (.195656)	.147491 (.196926)	.068569 (.199140)	-.056327 (.205432)	
SSR/25	.120555E+24	.920132	.865431	.869848	.895383	.919701	.934906	.823338	.535816E+22
SBIC	816.354	44.5248	43.6361	43.7099	44.1294	44.5180	44.7557	42.9131	771.209

Table 4.B
Sum of Squared Transformed Residuals, e’e
$Y_t \sim ARI(1,1)$ Process: $\Delta Y_t = \rho\, \Delta Y_{t-1} + v_t$

	ρ								
	- 5	- 1	- 0.5	-0.2	0	0.2	0.5	1	5
Integrated Series, AR1									
Interc.	.916647E+11 (.713744E+11)	-2.17192 (.420542)	-.606081 (.302965)	-.424343 (.258996)	-.389787 (.249213)	-.393963 (.251469)	-.472313 (.273739)	-1.20927 (.374808)	-.138889E+10 (.265188E+11)
AR coef.	-5.00000 (.926667E-08)	-.532373 (.156086)	.722075 (.125667)	.864003 (.087744)	.906446 (.071093)	.933177 (.058122)	.962455 (.041001)	1.06317 (.010178)	5.00000 (.338391E-08)
SSR/29	.154527E+24	4.05250	1.24542	.874182	.803366	.818064	.985140	2.27913	.207458E+23
SBIC	874.005	68.0771	49.7892	44.3031	42.9937	43.2747	46.1553	59.1562	842.880
Trend	.803719E+10 (.360081E+10)	-.160282 (.014680)	-.082443 (.021918)	-.057267 (.022030)	-.049381 (.022608)	-.046498 (.024088)	-.052265 (.028659)	-.158013 (.031223)	-.934399E+08 (.144560E+10)
AR coef.	-5.00000 (.898038E-08)	-.827417 (.104664)	.313337 (.174639)	.627745 (.143366)	.738632 (.123888)	.808855 (.106948)	.873778 (.082457)	1.00974 (.016287)	5.00000 (.354344E-08)
SSR/29	.139373E+24	1.52228	.952556	.774606	.748068	.786275	.974511	1.64472	.207448E+23
SBIC	872.405	52.9006	45.6340	42.4286	41.8883	42.6604	45.9872	54.0997	842.879
Interc.	-.195658E+12 (.143703E+12)	.684332 (.469293)	.283307 (.395799)	.095488 (.367327)	.00336283 (.366840)	-.079086 (.383593)	-.211269 (.454052)	1.09759 (.682758)	.601565E+09 (.581649E+11)
Trend	.172394E+11 (.763374E+10)	-.193115 (.026729)	-.099731 (.032741)	-.063636 (.033192)	-.049616 (.034487)	-.040649 (.037480)	-.034606 (.047795)	-.253910 (.066954)	-.122490E+09 (.317077E+10)
AR coef.	-5.00000 (.896157E-08)	-.842474 (.103207)	.276211 (.183604)	.613926 (.155121)	.738159 (.136223)	.819281 (.119942)	.897863 (.098323)	.984130 (.022479)	5.00000 (.370974E-08)
SSR/28	.135387E+24	1.46536	.968848	.800338	.774783	.813121	1.00157	1.55952	.214856E+23
SBIC	873.128	53.4830	47.0699	44.1083	43.6053	44.3539	47.5848	54.4483	844.596
Integrated Series, AR2									
Interc.	.947896E+11 (.751481E+11)	-.420150 (.266274)	-.411389 (.262303)	-.416746 (.265629)	-.419314 (.267195)	-.420620 (.267801)	-.420993 (.267964)	-.422746 (.267857)	-.143763E+10 (.279614E+11)
AR1 coef.	-5.00000 (.959795E-08)	-.076630 (.085262)	.251766 (.152900)	.581689 (.183053)	.826626 (.191448)	1.06012 (.189890)	1.37474 (.173280)	1.88750 (.118908)	5.00000 (.350997E-08)
AR2 coef.	-	.875600 (.085014)	.609484 (.149757)	.305768 (.179556)	.077637 (.188108)	-.137976 (.186927)	-.423308 (.171485)	-.886124 (.127410)	-
SSR/27	.165652E+24	.846045	.819970	.841349	.851404	.855175	.855989	.856254	.222825E+23
SBIC	848.052	43.5818	43.1122	43.4983	43.6765	43.7428	43.7571	43.7617	817.961
Trend	.804019E+10 (.373237E+10)	-.049405 (.023297)	-.045636 (.023838)	-.048092 (.024287)	-.050712 (.024555)	-.053654 (.024894)	-.058706 (.025838)	-.064666 (.026577)	-.934776E+08 (.149848E+10)
AR1 coef.	-5.00000 (.930685E-08)	-.237664 (.133710)	.158397 (.167946)	.507058 (.188658)	.761436 (.192495)	1.00379 (.187496)	1.32669 (.168163)	1.77368 (.129546)	5.00000 (.367243E-08)
AR2 coef.	-	.713975 (.133501)	.484016 (.170775)	.189388 (.189388)	-.031331 (.191171)	-.235569 (.184868)	-.489972 (.163144)	-.786956 (.132923)	-
SSR/27	.149687E+24	.792118	.787742	.801636	.802320	.796308	.784286	.767059	.222814E+23
SBIC	846.532	42.5939	42.5108	42.7730	42.7858	42.6730	42.4448	42.1117	817.960
Interc.	-.221128E+12 (.158819E+12)	-.00981954 (.404997)	-.076785 (.412741)	-.040252 (.423123)	.00873343 (.430142)	.070617 (.439368)	.208296 (.469686)	.590404 (.549513)	.659794E+09 (.646601E+11)
Trend	.184030E+11 (.829805E+10)	-.048729 (.036620)	-.040133 (.038270)	-.045147 (.039634)	-.051365 (.040721)	-.059101 (.042326)	-.076168 (.047313)	-.119621 (.057607)	-.125219E+09 (.346529E+10)
AR1 coef.	-5.00000 (.927696E-08)	-.236356 (.146547)	.164281 (.173931)	.508707 (.192998)	.761262 (.196348)	1.00359 (.190977)	1.32880 (.170789)	1.72798 (.136000)	5.00000 (.386287E-08)
AR2 coef.	-	.715309 (.146755)	.495618 (.184755)	.195572 (.203095)	-.032666 (.205605)	-.246102 (.199376)	-.517051 (.176524)	-.753789 (.136092)	-
SSR/26	.144658E+24	.822565	.816952	.832179	.833165	.826114	.808336	.762699	.231383E+23
SBIC	847.154	44.2941	44.1914	44.4684	44.4862	44.3587	44.0324	43.1607	819.661

Table 4.C
Sum of Squared Transformed Residuals, e'e
$Y_t \sim$ ARD(1,1) Process: $\Sigma Y_t = \rho\, \Sigma Y_{t-1} + v_t$

	ρ								
	- 5	- 1	- 0.5	-0.2	0	0.2	0.5	1	5
Differenced Series, AR1									
AR coef.	-5.00000 (.153698E-08)	-.986568 (.030839)	-.882429 (.095803)	-.763060 (.124151)	-.653837 (.142894)	-.529158 (.158924)	-.344754 (.174942)	-.089295 (.183821)	5.00000 (.102619E-07)
SSR/28	.900863E+22	1.90807	1.22427	1.10746	1.09914	1.09658	1.04815	.852216	.821202E+23
SBIC	775.335	51.6924	45.2581	43.8040	43.6946	43.6609	43.0059	40.0053	807.380
Interc.	.234966E+11 (.175864E+11)	.029349 (.261153)	.048239 (.209087)	.048881 (.198831)	.047626 (.198086)	.046471 (.197860)	.050367 (.193411)	-.126169 (.174891)	-.482933E+11 (.548672E+11)
AR coef.	-5.00000 (.153360E-08)	-.986558 (.031398)	-.882954 (.097491)	-.763754 (.126319)	-.654565 (.145393)	-.529899 (.161706)	-.345830 (.177976)	-.109588 (.187538)	5.00000 (.105807E-07)
SSR/27	.876293E+22	1.97782	1.26712	1.14591	1.13741	1.13488	1.08425	.867067	.827862E+23
SBIC	776.090	53.3693	46.9131	45.4552	45.3473	45.3150	44.6532	41.4121	808.653
Trend	.188618E+10 (.859302E+09)	-.763408E-04 (.013160)	.158330E-02 (.010545)	.172140E-02 (.010029)	.169035E-02 (.999056E-02)	.169640E-02 (.997893E-02)	.234496E-02 (.975667E-02)	-.499788E-02 (.878209E-02)	-.410564E+10 (.282238E+10)
AR coef.	-5.00000 (.148741E-08)	-.986569 (.031406)	-.883024 (.097601)	-.763943 (.126465)	-.654817 (.145555)	-.530256 (.161882)	-.347012 (.178209)	-.099384 (.186924)	5.00000 (.108035E-07)
SSR/27	.792761E+22	1.97874	1.26856	1.14722	1.13864	1.13598	1.08465	.873304	.789723E+23
SBIC	774.638	53.3760	46.9296	45.4718	45.3629	45.3290	44.6586	41.5160	807.969
Interc.	-.557263E+11 (.390951E+11)	.172926 (.630614)	.111238 (.505439)	.100885 (.480741)	.096977 (.478995)	.089884 (.478549)	.046240 (.468276)	-.200420 (.422687)	.129035E+12 (.128218E+12)
Trend	.448432E+10 (.200838E+10)	-.797653E-02 (.031770)	-.350170E-02 (.025478)	-.289060E-02 (.024233)	-.274303E-02 (.024145)	-.241301E-02 (.024124)	.229425E-03 (.023618)	.409651E-02 (.021149)	-.102799E+11 (.675296E+10)
AR coef.	-5.00000 (.148246E-08)	-.986588 (.031958)	-.882323 (.099419)	-.763008 (.128842)	-.653729 (.148308)	-.529030 (.164983)	-.345963 (.181881)	-.113261 (.191912)	5.00000 (.111887E-07)
SSR/26	.763582E+22	2.04892	1.31490	1.18933	1.18057	1.17807	1.12594	.899118	.789350E+23
SBIC	775.230	55.0178	48.5863	47.1310	47.0237	46.9930	46.3368	43.0748	809.099
Differenced Series, AR2									
AR1 coef.	-5.00000 (.159499E-08)	-1.61259 (.142383)	-1.10505 (.179245)	-.779235 (.188080)	-.600872 (.189151)	-.450508 (.189358)	-.263396 (.189755)	-.025809 (.186832)	5.00000 (.106493E-07)
AR2 coef.	-	-.636103 (.142503)	-.282236 (.183559)	-.054885 (.191920)	.035790 (.191958)	.083199 (.190686)	.118431 (.188643)	.229699 (.184724)	-
SSR/26	.970160E+22	1.06024	1.10127	1.09685	1.10109	1.10051	1.05305	.831600	.884371E+23
SBIC	750.797	42.8439	43.3754	43.3192	43.3732	43.3658	42.7486	39.4433	781.737
Interc.	.243563E+11 (.185851E+11)	-.357315E-02 (.198696)	-.442957E-02 (.202514)	-.497929E-02 (.202112)	-.456264E-02 (.202505)	-.330088E-02 (.202465)	.387363E-02 (.198155)	-.121299 (.178820)	-.501157E+11 (.580541E+11)
AR1 coef.	-5.00000 (.159251E-08)	-1.61246 (.145383)	-1.10486 (.182993)	-.779014 (.192012)	-.600666 (.193110)	-.450351 (.193346)	-.263631 (.193885)	-.042878 (.190471)	5.00000 (.110006E-07)
AR2 coef.	-	-.635971 (.145509)	-.282115 (.187274)	-.054789 (.195757)	.035849 (.195776)	.083230 (.194470)	.118373 (.192400)	.204167 (.190430)	-
SSR/25	.944107E+22	1.10263	1.14529	1.14070	1.14511	1.14452	1.09515	.849233	.893123E+23
SBIC	751.533	44.5098	45.0413	44.9849	45.0391	45.0318	44.4145	40.8541	782.992
Trend	.188900E+10 (.893560E+09)	.116491E-02 (.983843E-02)	.110412E-02 (.010027)	.110720E-02 (.010007)	.109486E-02 (.010026)	.111446E-02 (.010024)	.161696E-02 (.981386E-02)	-.350274E-02 (.883286E-02)	-.411228E+10 (.293528E+10)
AR1 coef.	-5.00000 (.154562E-08)	-1.61337 (.145311)	-1.10573 (.182855)	-.779938 (.191863)	-.601612 (.192970)	-.451390 (.193223)	-.265529 (.193840)	-.033643 (.190960)	5.00000 (.112278E-07)
AR2 coef.	-	-.636903 (.145442)	-.282506 (.187165)	-.054960 (.195674)	.035793 (.195713)	.083155 (.194414)	.117545 (.192350)	.217327 (.190367)	-
SSR/25	.855954E+22	1.10203	1.14476	1.14017	1.14459	1.14397	1.09398	.859457	.852793E+23
SBIC	750.161	44.5022	45.0347	44.9784	45.0327	45.0250	44.3995	41.0216	782.345
Interc.	-.628316E+11 (.435111E+11)	-.156586 (.505127)	-.155141 (.515249)	-.158929 (.514170)	-.154837 (.515122)	-.149107 (.514884)	-.162009 (.503269)	-.339546 (.446256)	.146207E+12 (.143292E+12)
Trend	.479933E+10 (.219707E+10)	.827102E-02 (.025018)	.814512E-02 (.025518)	.831986E-02 (.025464)	.812141E-02 (.025510)	.788035E-02 (.025498)	.897201E-02 (.024938)	.011729 (.021911)	-.110604E+11 (.741438E+10)
AR1 coef.	-5.00000 (.153916E-08)	-1.61240 (.148045)	-1.10352 (.186419)	-.777469 (.195594)	-.599394 (.196717)	-.449669 (.196954)	-.265384 (.197412)	-.047357 (.193430)	5.00000 (.116655E-07)
AR2 coef.	-	-.636010 (.148172)	-.279988 (.190848)	-.052373 (.199488)	.037834 (.199490)	.084261 (.198114)	.115931 (.195958)	.199656 (.193390)	-
SSR/24	.820343E+22	1.14337	1.18797	1.18296	1.18781	1.18748	1.13467	.874181	.851394E+23
SBIC	750.660	46.1123	46.6480	46.5889	46.6462	46.6423	46.0053	42.3540	783.417

Parameter: - 0.2

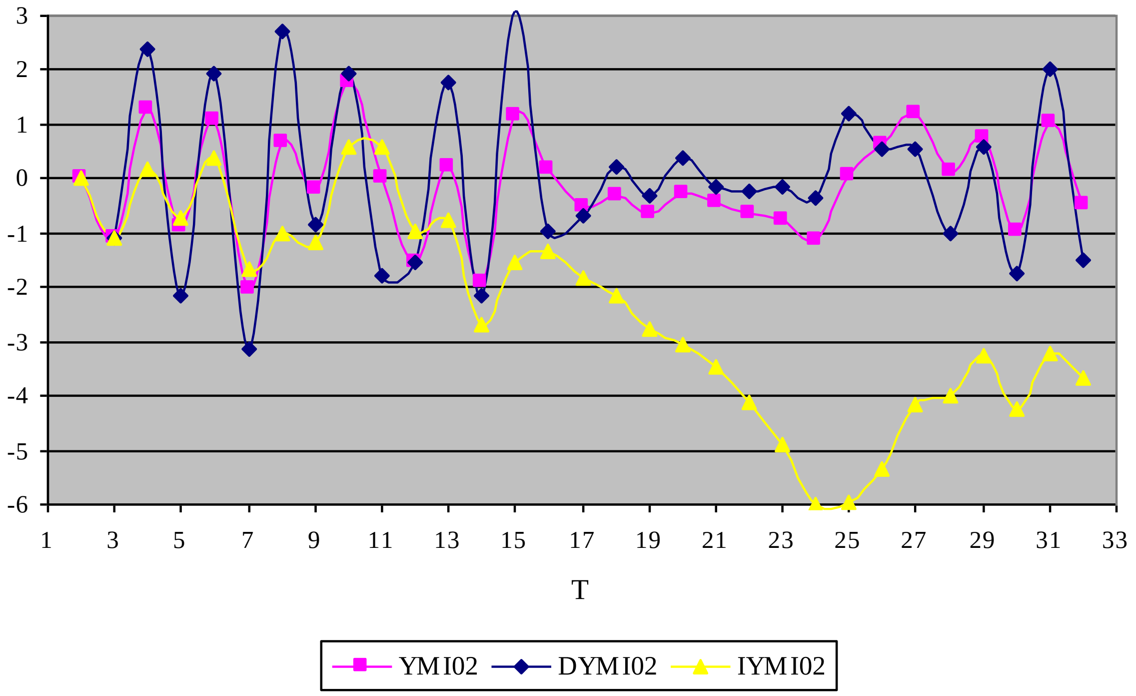

Fig. 1 – YM02 = AR(1); DYMI02 = ARD(1,1); IYMI02 = ARI(1,1)

Parameter: + 0.2

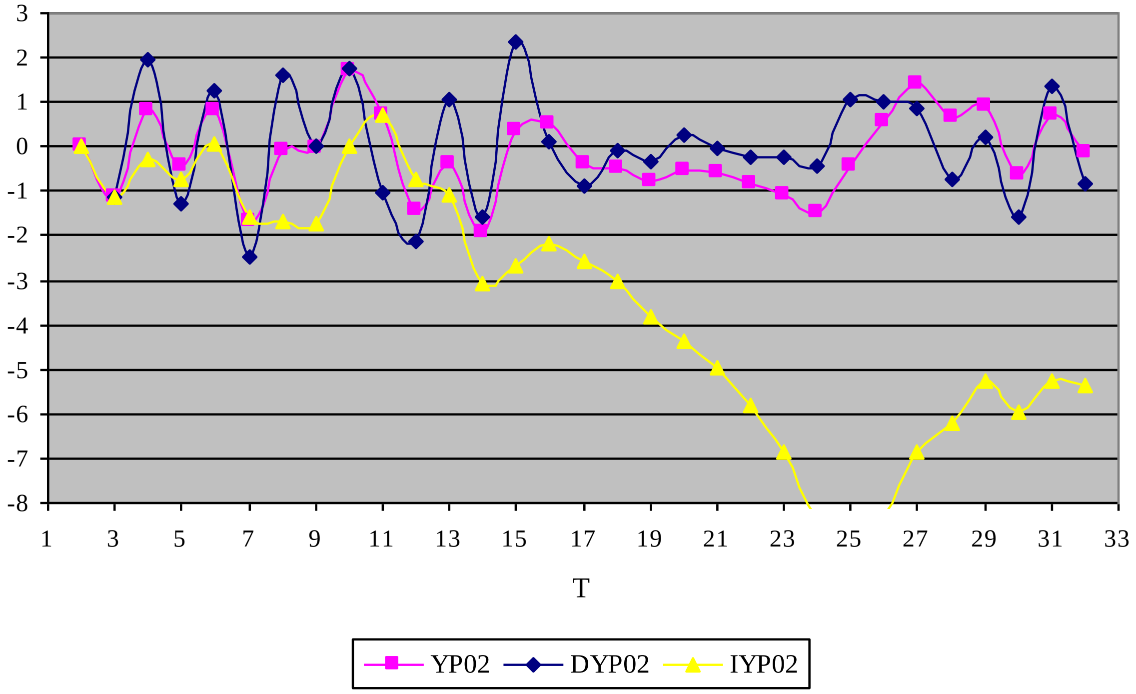

Fig. 2 – YP02 = AR(1); DYPI02 = ARD(1,1); IYPI02 = ARI(1,1)

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