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A low willingness to pay in a duopoly à la Hotelling: the role of the public firm.

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Abstract

The purpose of this paper is to analyze the role of the public firm in a spatial duopoly model *à la Hotelling* in the case of a low willingness to pay. We find that the presence of a public firm has the well known regulatory function in a market with a relative high willingness to pay; it is irrelevant in a market with a medium level of the willingness to pay; the relevance is for a low willingness to pay, where it ensures the full market coverage (as a result of the standard welfare maximization); finally, if the willingness to pay is very low, the public firm ensures a higher, but not full, market coverage with respect to the pure private case.

Finally, we find that, for a low willingness to pay, the presence of the public firm is not sufficient to guarantee the optimal market configuration, so that the efficient level of welfare.

Key words: mixed duopoly, full market coverage, low willingness to pay, efficient welfare

JEL codes:

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1 Introduction

Up until the 90s, the economic literature studied the interaction between public and private firms only in the case of homogenous goods. In a seminal paper, Cremer *et al* (1991) extended the analysis of mixed markets to a spatially differentiated framework *à la Hotelling*, finding that the maximum differentiation principle stated by d'Aspremont *et al* (1979) does not hold in the presence of a public firm. This result is due to the objective function of the latter: in a spatial framework with covered market welfare maximization amounts to minimizing consumers' transportation costs and this obviously implies closer locations at equilibrium. The lower product differentiation implies that price competition is stronger than in the private duopoly model.

The literature on spatial mixed oligopolies which followed Cremer *et al*'s contribution has investigated several extensions of the basic model – for example partial privatization as in Anderson *et al* (1997), and Lu and Poddar (2007); the existence of a leader-follower relation as in Matsumura and Matsushima (2003), and Sanjo (2009); the role of the demand elasticity as in Kitahara and Matsumura (2013); the role of the uniform distribution assumption in Benassi *et al* (2017). However, it has always preserved a very basic (but strong) assumption of the Hotelling model, namely that the willingness to pay is high enough to guarantee the full market coverage, at the equilibrium price, for all possible locations of the two firms.

In the case of private spatial models, the impact of this assumption has been thoroughly investigated. Economides (1984) and Hinloopen and van Marrewijk (1999) showed that in a private duopoly *à la Hotelling* with linear transportation costs, if the willingness to pay is low enough, full market coverage may not occur. The two private firms become independent monopolists with an unserved portion of the market, due to undetermined locations. Economides (1989) and Chirco *et al* (2003) extended the analysis to the case of quadratic costs in a Salop circular city and a Hotelling linear city, respectively. In particular, Chirco *et al* (2003) show the role played by the assumption of a 'high enough' willingness to pay in terms of the principle of maximum differentiation: a high willingness to pay is the condition which allows firms to strongly differentiate their products and impose high transportation costs, in order to relax price competition.

The purpose of this paper is to study the impact of a low willingness to pay in a mixed spatial duopoly model, and to compare the results with those obtained by Chirco *et al* (2003) in the traditional fully private model. The motivation to investigate the role of a public firm in a spatial model with low demand goes beyond the analytics: in the political debate and in the common wisdom, a key role of public firms is to facilitate the access to the market to a broad range of consumers: social care, education, social housing, access to water are all examples of markets of basic necessities in which the coexistence of public firms with private firms is seen as an instrument to reach most consumers at affordable prices.

In order to facilitate comparison with Chirco *et al* (2003), we shall adopt the same market configuration, by studing a mixed spatial duopoly where a private (profit-maximizing) and a public (welfare-maximizing) firm compete along a linear city in the presence of quadratic transportation costs. The analysis will highlight several peculiar aspects of the mixed market setup: the extent to which the public firm perceives an incentive to guarantee full market coverage, the role played by the private firm, and the welfare distortion with respect to the social optimum which arises in the presence of low willingness to pay.

There are two main results of the paper. The first is an irrelevance result: there is a range of the willingness to pay at which the presence of a public agent is absolutely irrelevant and the mixed market behaves exactly in the same way as a fully private market. The second result is that, in another range of the willingness to pay, full market coverage occurs in a mixed market, whereas in the private duopoly this outcome is not guaranteed.

The paper is organized as follows. In Section 2 the standard mixed duopoly model à la Hotelling is briefly resumed. In Section 3 the role of a low willingness to pay in a mixed duopoly is analyzed and compared with the private duopoly case. Some conclusions are gathered in Section 4.

2 The standard mixed duopoly model

We consider a spatial mixed duopoly *à la Hotelling*, as in Cremer *et al* (1991). There are two technologically identical firms, 1 and 2, producing a homogeneous good at constant unit and marginal costs, normalized to zero (without any loss of generality). Firm 1 is a profit-maximizing private firm, whereas firm 2 is a welfare-maximizing public firm.

Firms interact strategically in a two-stage sequential game. In the first stage, firms choose simultaneously their location x_i, $i = 1, 2$ in the unit interval $[0, 1]$; in the second stage, they set their prices p_i. Let $x_1 < x_2$ by assumption.

Consumers are uniformly distributed with density 1 along the interval $[0, 1]$ and consume at most one indivisible unit of the good. Consumption entails a gross surplus s, and a disutility given by the price p_i and a transportation cost which is assumed to be quadratic in distance. Therefore, the net utility accruing to a consumer located in $a \in [0, 1]$ is given by

$$U(a) = s - p_i - t(x_i - a)^2 \qquad i = 1, 2$$

if she buys good i, which occurs if $s - p_i - t(x_i - a)^2 \geq 0$; if, on the contrary, $s - p_i - t(x_i - a)^2 < 0$ for all i, the consumer does not buy and $U(a) = 0$. The gross surplus s can therefore be interpreted as the consumers' willingness to pay or the gross reservation price.

Assume that s is high enough to ensure that all consumers buy one unit of the good. Then, between the two firms there is a consumer, located at $\tilde{a} \in [x_1, x_2]$, who is indifferent between the two goods, so that:

$$s - p_1 - t(x_1 - \tilde{a})^2 = s - p_2 - t(x_2 - \tilde{a})^2. \tag{1}$$

The location $\tilde{a}$ defines the market share of two firms, $y_1 = \tilde{a}$ and $y_2 = 1 - \tilde{a}$. From (1) we can write the demand functions as follows:

$$\begin{aligned} y_1 &= \tilde{a} = \frac{p_2 - p_1 + t(x_2^2 - x_1^2)}{2t(x_2 - x_1)} \\ y_2 &= 1 - \frac{p_2 - p_1 + t(x_2^2 - x_1^2)}{2t(x_2 - x_1)} \end{aligned}$$

2.1 Price equilibrium

The price-location game is solved by backward induction. Given locations, at the price stage of the game, firm 1 maximizes its profit π_1 with respect to its price:

$$\max_{p_1} \pi_1 = p_1 \frac{p_2 - p_1 + t(x_2^2 - x_1^2)}{2t(x_2 - x_1)}$$

whereas firm 2 maximizes welfare, which amounts, under full market coverage, to minimizing the total transportation costs:

$$\max_{p_2} W = s - \int_0^{\tilde{a}} t(x_1 - a)^2 da - \int_{\tilde{a}}^1 t(x_2 - a)^2 da.$$

The best reply functions are, respectively:

$$p_1(p_2) = \frac{p_2 + t(x_2 - x_1)}{2} \tag{2}$$

and

$$p_2(p_1) = p_1. \tag{3}$$

Using (2) and (3) we obtain the price equilibrium, which is the same for both firms:

$$p_1 = p_2 = p^E = t(x_2^2 - x_1^2)$$

so that the location of the indifferent consumer (and then firm 1's market share) becomes the mean between the two locations:

$$\tilde{a} = \frac{x_1 + x_2}{2}.$$

2.2 Location equilibrium

Given the solution of the price stage, maximization of the firms' objective function with respect to their locations yields the following best reply functions:

$$x_1 = \frac{x_2}{3}$$
$$x_2 = \frac{1}{3}x_1 + \frac{2}{3}$$

so that the equilibrium locations are:

$$\left(x_1^E, x_2^E\right) = \left(\frac{1}{4}, \frac{3}{4}\right)$$

and the equilibrium price is:

$$p_1 = p_2 = p^* = \frac{1}{2}t.$$

When compared with the standard private spatial duopoly *à la Hotelling,*[1] the mixed duopoly equilibrium exhibits lower product differentiation and lower prices.

3 The role of the willingness to pay: the private vs the mixed spatial duopoly

The above result relies upon the assumption that the willingness to pay of consumers is *high enough* to ensure the full market coverage. The implications of relaxing this assumption in a private spatial duopoly with quadratic costs has been investigated by Chirco *et al* (2003). If the gross consumers' surplus is not high enough to ensure that all consumers are served at the equilibrium price for any location − where high enough means $s \geq 33t/16$ − the incentive perceived to maximum differentiation is softened by a demand incentive to reach the consumers located at the centre of the linear city.

[1] In d'Aspremont *et al* (1979) the locations are $\{-1/4; 5/4\}$ and the equilibrium price is $p_1 = p_2 = p^* = 3/2t$.

In particular, Chirco *et al* (2003) show that for $s \in [9t/16; 33t/16]$, firms react to a decrease in the willingness to pay by progressively moving inwards (from $\{-1/4; 5/4\}$ to $\{1/4; 3/4\}$), thus reducing their optimal price. As s decreases, there is a reduction in the degree of product differentiation, each firm maintaining its market share equal to 1/2, but enjoying a lower profit. Within this interval the relocation of firms implies that the price reduction is less than proportional to the decrease in the willingness to pay. For $s \in [3t/16; 9t/16]$, any further reduction in s is not accompanied by changes in locations: firms keep on setting in $\{1/4; 3/4\}$ and reduce linearly the price in order to fully cover the market. Chirco *et al* (2003) call these equilibriums 'kink'. Finally, if $s \in [0; 3t/16]$ firms become independent monopolists, locations are undefined and partial market coverage occurs at equilibrium: in this range s is so low that any attempt to preserve firms' market shares and extensive margins through further price reductions is dominated by the choice of deepening the intensive margin on served consumers.

The introduction of a willingness to pay constraint in a mixed market exhibits several differences with respect to the fully private case. First of all, the unconstrained equilibrium is different: maximum differentiation in the private duopoly, the socially efficient differentiation (and a lower price) in the mixed case. Second, in a mixed market the competing firms have different objective functions, and therefore potentially asymmetric reactions to the demand constraint. Finally, there is a firm, the public one, whose objective function embodies the degree of market coverage.

3.1 The irrelevance of the public firm

The equilibrium of the unconstrained mixed duopoly, described in Section 2, ensures that the willigness to pay constraint becomes actually binding for values of $s \leq 9t/16$. At the unconstrained equilibrium, this is actually the total disutility (in terms of money) of the consumers travelling the maximum distance in order to be served by one of the firms.

Consider now the range $s \in [3t/16; 9t/16]$. For this range of the willingness to pay we establish the following Proposition.

Proposition 1 *For $s \in [3t/16; 9t/16]$, the equilibrium prices and locations in a mixed spatial duopoly coincide with the equilibrium prices and locations in a private spatial duopoly, with $x_1 = 1/4$, $x_2 = 3/4$ and $p_1 = p_2 = s - t/16$.*

Proof. Proving Proposition 1 amounts to verifying that if firms are located in $x_1 = 1/4$ and $x_2 = 3/4$, and set $p_1 = p_2 = s - t/16$, neither the private nor the public firm perceive an incentive to deviate from their choice at the price or at the location stage. We proceed in two steps. In Step I we prove that the candidate kink equilibrium is deviation-proof at the price stage, in Step II that it is deviation-proof at the location stage.

Step I. First of all notice that given $x_1 = 1/4$ and $x_2 = 3/4$, the price $p_1 = p_2 = s - t/16$ is the maximum symmetric price at which the whole market is served, with the consumers located in 0, 1/2 and 1 enjoying a null net surplus, the profit of the private firm being $\pi_1^* = 1/2\,(s - t/16)$, and welfare being $W = s - t/48$.

Consider now the private firm. It is easy to check that it has no incentive to set a price lower than $(s - t/16)$. Indeed, given the price set by its rival, if it reduced its price by an amount ε, the indifferent consumer would be located in $(t + 2\varepsilon)/2t$ and its profit would be $\pi_\varepsilon = (s - t/16 - \varepsilon)(t + 2\varepsilon)/2t$ which is lower than π_1^* for all $\varepsilon > 0$ in the relevant interval. Moreover, the private firm has no incentive to set a price higher than $(s - t/16)$. If it did so, it would separate from the public firm, leaving an unserved market share both in the centre and in the neighborhood of 0 of the linear city. Given the price of the public firm, at $p_1 = s - t/16$ the elasticity of the demand of the private firm to a price increase is $8s - 1/2$, which is higher than 1 for $s > 3t/16$; therefore, any increase in the price of the private firm would lead to a reduction in revenues and profits.

As far as the public firm is concerned, it has no incentive to reduce its price because once the market is fully covered, any asymmetry in prices generates a welfare loss due to an increase in total transportation costs. On the other hand, it has also no incentive to increase its price, because, given the choice of its rival, if it raised its price by setting it at a level $\overline{p}_2$ (at which some consumers in the right neighborhood of 1/2 and at the left neighborhood of 1 are unserved) total welfare would be written as in equation 4:

$$W(\overline{p}_2) = s\left(\frac{1}{2} + 2\sqrt{\frac{s - \overline{p}_2}{t}}\right) - \int_0^{\frac{1}{2}} t\left(x - \frac{1}{4}\right)^2 dx - \int_{\frac{3}{4}-\sqrt{\frac{s-\overline{p}_2}{t}}}^{\frac{3}{4}+\sqrt{\frac{s-\overline{p}_2}{t}}} t\left(x - \frac{3}{4}\right)^2 dx \tag{4}$$

which is unambiguously decreasing in $\overline{p}_2$.

Step II. We turn now to the location stage. Consider first the private firm.

When it evaluates the incentive to change its location, it must anticipate both firms' optimal behavior at the price stage. Should it move outwards and set a price equal or lower than the kink price, the public firm would mimic this price in order to minimize the transportation costs: the consumer indifferent between the two firms would move to the left of 1/2, and the reduction in price would be accompanied by a reduction of the private firm's market share. The result is clearly a lower profit. On the other hand, moving outwards and setting a price higher than the kink price is also clearly unprofitable. Assume that the private firm locates at the left of 1/4 in a generic location x_1^l, and sets a generic price $\widetilde{p}_1$ higher than the kink price. Depending on x_1^l and $\widetilde{p}_1$ the consumers in the neighbourhood of 0 may be fully served or unserved. Assume first that some consumers close to zero are unserved. The best reaction of the public firm would be to set the maximum price at which there are no unserved consumers between the two firms. Indeed, in this case, welfare would be:

$$W = s\left(2\sqrt{\tfrac{s-\widetilde{p}_1}{t}} + 2\sqrt{\tfrac{s-\bar{p}_2}{t}}\right) - \int_{x_1^l-\sqrt{\frac{s-\widetilde{p}_1}{t}}}^{x_1^l+\sqrt{\frac{s-\widetilde{p}_1}{t}}} t\left(x - x_1^l\right)^2 dx +$$
$$-\int_{\frac{3}{4}-\sqrt{\frac{s-\bar{p}_2}{t}}}^{\frac{3}{4}+\sqrt{\frac{s-\bar{p}_2}{t}}} t\left(x-\tfrac{3}{4}\right)^2 dx \qquad \text{for } \bar{p}_2 > s - \tfrac{1}{16}t;$$

$$W = s\left(2\sqrt{\tfrac{s-\widetilde{p}_1}{t}} + \left(\sqrt{\tfrac{s-p_2}{t}} + \tfrac{1}{4}\right)\right) - \int_{x_1^l-\sqrt{\frac{s-\widetilde{p}_1}{t}}}^{x_1^l+\sqrt{\frac{s-\widetilde{p}_1}{t}}} t\left(x - x_1^l\right)^2 dx +$$
$$-\int_{\frac{3}{4}-\sqrt{\frac{s-\bar{p}_2}{t}}}^{1} t\left(x-\tfrac{3}{4}\right)^2 dx \qquad \text{for } s - \tfrac{1}{16}t \geq p_2 \geq \beta;$$

$$W = s\left(1 - \left(x_1^l - \sqrt{\tfrac{s-\widetilde{p}_1}{t}}\right)\right) - \int_{x_1^l-\sqrt{\frac{s-\widetilde{p}_1}{t}}}^{\frac{p_2-\widetilde{p}_1-t\left(x_1^l\right)^2+t\left(\frac{3}{4}\right)^2}{2t\left(\frac{3}{4}-x_1^l\right)}} t\left(x - x_1^l\right)^2 dx +$$
$$-\int_{\frac{p_2-\widetilde{p}_1-t\left(x_1^l\right)^2+t\left(\frac{3}{4}\right)^2}{2t\left(\frac{3}{4}-x_1^l\right)}}^{1} t\left(x-\tfrac{3}{4}\right)^2 dx \qquad \text{for } p_2 < \beta;$$

with $\beta = s - t\left(\frac{3}{4} - \left(x_1^l + \sqrt{\frac{s-\widetilde{p}_1}{t}}\right)\right)^2$.

It can be checked that for $p_2 > \beta$ the welfare is decreasing in p_2, whereas for $p_2 < \beta$ it is increasing in p_2 (given that $p_1 > p_2$).Therefore, the best reaction of the public firm would be to set $p_2 = \beta$ which is the maximum price which allows to serve all consumers located between the two firms.

It is easy to check that the same would apply in case the private firm set a price such that the market at its left would be fully covered.

Hence, in case the private firm relocated at the left of $1/4$ and increased its price, the public firm would not compete for intermediate customers, but it would simply cover the market between the two firms. The private firm could choose its price as if it faced no competition from the public firm. But if in absence of any reaction by the public firm, setting any price higher than $s - t/16$ was not profitable when the private firm was located in $1/4$, it cannot be profitable when the private firm is located at the left of $1/4$, with a market share which at best is the same it could gain when located in $1/4$.

Alternatively, the private firm could move inwards and lower its price, in order to increase its market share. Assume that the private firm sets in a location $1/4 + \epsilon$ and lowers its price to a generic level $\underline{p}_1 < s - t/16$. The public firm would react by reducing its price to $p_2 = \underline{p}_1$ so that the consumer indifferent between the two firms would be anyhow located in $1/2 + \epsilon/2$. There are two possibilities. The first is that the private firm lowers its price up to the maximum level ensuring full market coverage, i.e. $\underline{\underline{p}}_1 = s - t\,(1/4 + \epsilon)^2$. Profits then would be:

$$\underline{\pi}_1(\epsilon) = \left(s - t\left(\frac{1}{4}+\epsilon\right)^2\right)\left(\frac{1}{2}+\frac{1}{2}\epsilon\right)$$

which is certainly lower than the kink profits if $s < 9t/16$. The alternative possibility is that $s - t(1/4+\epsilon)^2 < \underline{p}_1 < s - t/16$. In this case some consumers are unserved in the neighbourhood of 0 and the profit function can be written as:

$$\underline{\pi}_1\left(\underline{p}_1, \epsilon\right) = \underline{p}_1\left(\frac{1}{2}+\frac{1}{2}\epsilon - \left(\frac{1}{4}+\epsilon-\sqrt{\frac{s-\underline{p}_1}{t}}\right)\right)$$

Notice that in this configuration, the market share of firm 1 at its left is larger than the market share at its right, where it faces the competition of the public firm. Therefore, $\underline{\pi}_1\left(\underline{p}_1, \epsilon\right)$ is certainly lower than the profit that the private firm could earn if it set $\underline{p}_1$ in the absence of a competing public firm:

$$\underline{\pi}_1\left(\underline{p}_1, \epsilon\right) < 2\underline{p}_1\sqrt{\frac{s-\underline{p}_1}{t}}$$

Moreover, the maximum value of the RHS of the above expression is the monopoly profit $\pi^M = (8s/9)\sqrt{3s/t}$ which is lower than the profits at the kink candidate equilibrium if $s > 3t/16$. Therefore, there is no incentive for the private firm to move from 1/4 inwards and lower its price.

Finally consider the incentives of the public firm at the location stage. Given that the private firm is located in 1/4, the public firm perceives no incentive to move from 3/4. Indeed, the kink equilibrium is a socially optimal configuration: the whole market is covered and the transportation costs are at their minimum possible level. Any different configuration would imply a welfare loss. ■

Therefore, by comparing results in the private and the mixed model, for a high (but not 'enough') willingness to pay, we can draw the following Conclusion:

Conclusion 1 *Whereas the public firm has a regulatory function in the range* $s \in [9t/16; 33t/16]$ – *the price and the product differentiation are lower than in the private duopoly model* – *in the range* $s \in [3t/16; 9t/16]$ *the presence of the public firm is irrelevant. Indeed, the same locations and equilibrium prices are observed under a private and a mixed market structure.*

3.2 The relevance of the public firm for market coverage

For a significantly low willingness to pay ($s < 3t/16$), Chirco *et al* (2003) show that, in the private duopoly model, firms set the monopoly price, accept a market share lower than 1/2, locations are undefined and partial market coverage obtains in equilibrium. Locations' ranges are $x_1 \in \left[\sqrt{s/(3t)}; 1/2 - \sqrt{s/(3t)}\right]$ and $x_2 \in \left[1/2 + \sqrt{s/(3t)}; 1 - \sqrt{s/(3t)}\right]$.

In the above paragraph we have demonstrated that if the private firm is located in 1/4 it perceives no incentive to raise its price above the kink price in order to separate from the public firm. However this incentive emerges for $s < 3t/16$, when the elasticity of demand with respect to a price increase becomes lower than 1, and the monopoly profits with simmetric market shares at the left and at the right of the private firm location become higher than the kink profits.

On the basis of this observation, we state the following Proposition:

Proposition 2 *For* $s \in \left[3t/(16 + 8\sqrt{3}); 3t/16\right[$, *in a mixed spatial duopoly* *the private price is the monopoly one* $p_1^M = 2s/3$ *and its location is* $x_1 \in \left[\sqrt{s/(3t)}; 1/2 - \sqrt{s/(3t)}\right]$, *whereas the public set a different price* $p_2 = s - t\left(1/2\left(1 + x_1 + \sqrt{s/(3t)}\right)\right)^2$ *and its location is* $x_2 = 1/2\left(1 + x_1 + \sqrt{s/(3t)}\right)$.

Before proving the above Proposition, a few observations are in order. Notice that once the private firm sets the monopoly price, there could be segments of the market line which are unserved, and this in turn implies a range of undefined locations for the private firm itself. In the sequel, and in particular in the Proof, we shall assume that a private firm behaving as an independent monopolist chooses, among all the locations ensuring monopoly profits, the location which guarantees symmetric market shares and full coverage in the neighbourhood of 0, i.e. $\sqrt{s/(3t)}$.[2] This location leaves the whole unserved market exclusively in the center of the linear city, allowing in principle the public firm to cover it. Indeed, the marginal consumer is located in $2\sqrt{s/(3t)}$ and the public firm can choose the location $x_2 = 1/2 + \sqrt{s/(3t)}$, which is halfway between the marginal consumer and the consumer located at 1. In this way the public firm guarantees the full coverage of the market by setting the price $p_2 = s - t\left(1/2 - \sqrt{s/(3t)}\right)^2$, which leaves consumers located in $2\sqrt{s/(3t)}$ and 1 with null net surplus.[3] We can now proceed to the Proof of Proposition 2.

Proof. Proving Proposition 2 amounts to verify that in the mentioned range $s \in \left[3t/(16 + 8\sqrt{3}); 3t/16\right[$ the private firm has the incentive to set the monopoly

[2] A possible justification is to assume that the government, interested in market coverage, pays an arbitrarily small incentive to the private firm in order to make it choose the location $\sqrt{s/(3t)}$.

[3] This market configuration implies symmetric market shares as in the same range of s of Chirco *et al* (2003).

price as in the private duopoly, and that the public firm has the incentive to guarantee the full coverage of the market between the two firms. As in the case of Proposition 1, we proceed in two steps. In Step I we prove that the candidate equilibrium is deviation-proof at the price stage, in Step II that it is deviation-proof at the location stage.

STEP I. At the price stage, we already know that if $s < 3t/16$ the private firm perceives the incentive to separate from the public firm. Once its location allows for symmetric market shares at its left and its right, the most profitable separating price is obviously the monopoly price $p_1^M = 2s/3$.

From the public firm's point of view, once the private firm is located in $\sqrt{s/(3t)}$, then a price lower than $p_2 = s - t\left(1/2 - \sqrt{s/(3t)}\right)^2$ would not imply advantages in terms of market coverage, but it would rather impose higher transportation costs to the indifferent consumer leading to higher overall transportation costs.

On the other hand, a higher price $\overline{p}_2$ would leave a segment of the market uncovered between the two firms. Welfare would be given by Equation 5:

$$\begin{aligned} \tilde{W}\left(p_2\right) = s\left(2\sqrt{\tfrac{s}{3t}} + 2\sqrt{\tfrac{s-p_2}{t}}\right) - \int_0^{2\sqrt{\frac{s}{3t}}} t\left(x - \sqrt{\tfrac{s}{3t}}\right)^2 dx + \\ - \int_{\frac{1}{2}+\sqrt{\frac{s}{3t}}-\sqrt{\frac{s-p_2}{t}}}^{\frac{1}{2}+\sqrt{\frac{s}{3t}}+\sqrt{\frac{s-p_2}{t}}} t\left(x - \left(\tfrac{1}{2} + \sqrt{\tfrac{s}{3t}}\right)\right)^2 dx. \end{aligned} \tag{5}$$

By deriving Equation 5 with respect to p_2 it turns out that $(\partial\tilde{W})/(\partial p_2) = -p_2(\sqrt{(s-p_2)/t})/(s-p_2) < 0$ if $p_2 > 0$.

Hence, the public firm avoids to be too aggressive and sets the maximum (positive) price that allows it to cover the residual market leaved unserved by the private one (the argument is similar to that made in Step I of the Proof of Proposition 1).

STEP II. At the location stage, if we assume that within the location interval $\left[\sqrt{s/(3t)}; 1/2 - \sqrt{s/(3t)}\right]$ the private firm locates at $\sqrt{s/(3t)}$, it has no incentive to move inwards or outwards, because, given the location of the public firm, this would reduce its market share unless it reduced its price below the monopoly price.

As far as the public firm is concerned, we know that, for a given gross welfare, it just tries to minimize transportation costs. The latter are clearly minimized with a symmetric configuration (see the Appendix). Given the location of the private firm, $x_2 = 1/2 + \sqrt{s/(3t)}$ is the location that allows symmetric transportation costs towards the public firm; therefore the public firm has no incentive to move inwards or outwards (regardless of any subsequent price adjustment). Thus the public firm has no incentive to deviate from the candidate equilibrium. ■

As shown by equation (5), the equilibrium configuration is sustainable as long as the public firm can set a non-negative price to guarantee the full coverage. Hence, the condition of non-negative price by the public firm defines the threshold $s = 3t/(16 + 8\sqrt{3})$, below which the public firm should set a negative price (that we can see as a subsidy) to cover the residual market. We can therefore conclude that, in the range $s \in \left[3t/(16 + 8\sqrt{3}); 3t/16\right[$, welfare maximization leads the public firm to guarantee, in its interaction with the private firm, the full market coverage as an implicit goal of the welfare maximization.

Finally, we can state the following Corollary of Proposition 2 which explains the equilibrium configuration for the lowest range of the willingness to pay:

Corollary 1 *If* $s < 3t/(16+8\sqrt{3})$ *the private firm is located in* $\sqrt{s/(3t)}$ *and sets the monopoly price* $p_1^M = 2s/3$*, whereas the public firm sets a null price, which leaves a portion of the market unserved. The two firms are independent (private and public monopolist), transportation costs are symmetric and the public firm has a range of indifferent locations* $x_2 \in \left[2\sqrt{s/(3t)} + \sqrt{s/t}; 1 - \sqrt{s/t}\right]$.

Hence the public firm covers the residual market leaved unserved by the private firm as long as it can set a non-negative price, because a negative price would imply a welfare reduction. The public locations' range depends on the incentivized private location $\sqrt{s/(3t)}$. For any different private location in the interval $\left[\sqrt{s/(3t)}; 1/2 - \sqrt{s/(3t)}\right]$, there would be a stable equilibrium, because the public firm would not compete for the private market share, but $\sqrt{s/(3t)}$ is the only location which allows the full market coverage. Furthermore, for any different private location, the value of the willingness to pay at which the public firm would set a null price would be lower than $s = 3t/(16 + 8\sqrt{3})$, because of the smaller market share to be covered.

By comparing results in the private and the mixed model, for a low willingness to pay, we can conclude that:

Conclusion 2 *If* $x_1 = \sqrt{s/(3t)}$*, in the range* $s \in \left[3t/(16 + 8\sqrt{3}); 3t/16\right[$ *the presence of the public firm guarantees the full market coverage, whereas in the range* $s \in \left[0; 3t/(16 + 8\sqrt{3})\right[$ *in both models there is an unserved market share,* *though the null public price ensures a higher market coverage with respect to the private model.*

3.3 The distortion from social optimum

In our spatial setup with low demand, with full market coverage the social optimum W^* is defined by the pair of qualities $\{1/4, 3/4\}$, while the price is undetermined in the range $[0; s - t/16]$.The upper bound of the optimal price range implies that if $s < t/16$ the optimal prices for the two products is zero, the full market coverage does not occur and welfare progressively goes to zero as s decreases. Therefore the socially optimal welfare is:

$$\begin{array}{ll} W^* = s - \frac{1}{48}t & \text{if } \; s \geq \frac{1}{16}t \\ W^* = \frac{8}{3}s\sqrt{\frac{s}{t}} & \text{if } \; s \leq \frac{1}{16}t. \end{array}$$

If $s \geq 3t/16$, welfare at the mixed market equilibrium, W_M, coincides with the socially optimal welfare, because the locations are indeed $\{1/4; 3/4\}$ and the price is in the aforementioned range. On the contrary, if $s < 3t/16$ we have to distinguish the range of s consistent with full market coverage from the range at which full coverage does not occur:

$$\begin{array}{ll} W_M = W^* = s - \frac{1}{48}t & \text{if } \; s \geq \frac{3}{16}t \\ W_M = \frac{2}{3}s - \frac{1}{12}t + \frac{1}{6}\sqrt{3st} & \text{if } \; s \in \left[\frac{3t}{16+8\sqrt{3}}; \frac{3}{16}t\right] \\ W_M = \frac{4}{27}s\sqrt{\frac{s}{t}}\left(4\sqrt{3}+9\right) & \text{if } \; s \in \left[0; \frac{3t}{16+8\sqrt{3}}\right]. \end{array}$$

In the private model, the equilibrium configuration is efficient only in the kink equilibria. Indeed, if $s \geq 33t/16$ product differentiation is too high and equilibrium is inefficient. Denoting with W_P welfare at the private duopoly equilibrium we have:

$$W_P = s - \frac{13}{48}t \quad \text{if } \; s \geq \frac{33}{16}t \; .$$

In the range $s \in [9t/16; 33t/16[$, by lowering the willingness to pay, the firms move progressively inwards, until they reach the optimum at the kink range:

$$\begin{array}{ll} W_P = s - \frac{31}{12}t + \frac{5}{2}t\sqrt{1+s} - ts & \text{if} \quad s \in \left[\frac{9}{16}t; \frac{33}{16}t\right] \\ W_P = W^* = s - \frac{1}{48}t & \text{if} \quad s \in \left[\frac{3}{16}t; \frac{9}{16}t\right] . \end{array}$$

Finally, if $s < 3t/16$ the two private firms become independent monopolist and the equilibrium configuration is again suboptimal:

$$W_P = \frac{32}{27}s\sqrt{\frac{3s}{t}} \quad \text{if} \quad s < \frac{3}{16}t \ .$$

By comparing the mixed and the private equilibrium configuration with the optimal one, it is possible to compute the welfare distortion. This is done in Figure 1, drawn under the assumption $t = 1$. The latter exhibits several interesting features. As far as the private market is concerned, it is worth stressing that in the range $s \in [9/16, 33/16]$ a decrease in the willingness to pay causes a reduction in the distorsion with respect to the social optimum: as the market becomes poorer it becomes more efficient. This process leads to an efficient outcome in the range $s \in [3/16, 9/16]$; however, when the market becomes very poor there is again a distorsion, due to the incentive perceived by the private firms to separate and become private monopolists. On the contrary, the mixed market is always efficient with exception of the case $s < 3/16$. Notwithstanding the presence of a public firm, the latter is not able to prevent that the private firm behaved as a monopolist. Clearly the distorsion is the lower than in the fully private duopoly.

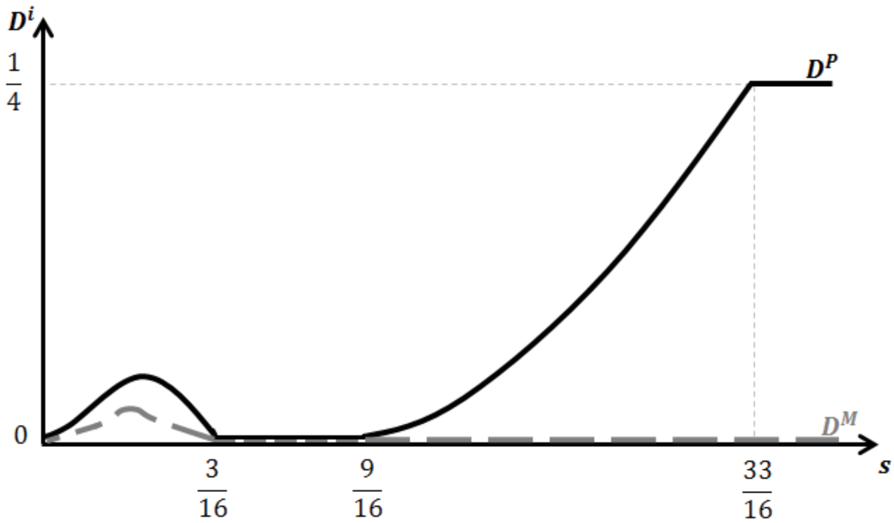

Figure 1: Welfare distortion (D^P: private model; D^M: mixed model).

4 Conclusions

In this paper, we have investigated the effect of a low willingness to pay in a spatial mixed duopoly *à la Hotelling*, by comparing its outcomes with the equivalent model in the pure private case.

The analysis starts by relaxing the standard assumption of a willingness to pay 'high enough' to guarantee the full market coverage. This 'high enough' assumption is strong, because it eliminates the effect of the demand constraint.

From Chirco *et al* (2003) we know that in the private model, by lowering the willingness to pay (starting from the 'high enough' value), at first there is a reduction in price and product differentiation. Then, there is just a linear reduction of the price (kink equilibra). Finally, the two firms become indipendent monopolists, locations are undefined and a portion of the market is unserved.

In a mixed duopoly model, we find that the two firms exhibit closer locations and a lower price for medium-high values of the willingness to pay with respect to the pure private case. Then, we show that in the kink equilibria there is exactly the same market configuration in both models, so that the presence of the public firm is irrelevant. Finally, if the willingness to pay is low, the public firm ensures the full market coverage by setting a price lower than the that of the private firm. Finally, when the willingness to pay is very low, the public firm sets a null price and the full market coverage may not occur even in the mixed case.

We can conclude that in rich markets (i.e. with a high willingness to pay), the presence of a public firm leads to a stronger competition which implies lower prices, lower product differentiation but also lower transportation costs. The net result is clearly a higher welfare (at the optimal level) and the well known regulatory function of the public firm.

In the kink range, the presence of a public firm is irrelevant and the configuration of the two markets is exactly the same. In the private model, the strategic effect – which induces firms to move outwards – is counterbalanced (with the same strength) by the demand effect. Whereas in the mixed model the incentive of the private firm to move outwards is counterbalanced by the incentive of the public firm to minimize transportation costs (and thus to move inwards). Both models show a socially efficient welfare: in this range the presence of a public firm is irrelevant.

Finally, in poor markets, the presence of a public firm always leads to a higher market coverage and a higher welfare. However, it does not guarantee the full coverage if a negative price is required, because this would lead to a lower welfare. Therefore, there is a threshold of the willingness to pay below which it sets a null price that guarantees a higher coverage, but not the full coverage of the market. We also show that, in this range, neither in the mixed nor in the private duopoly the market outcome is efficient.

We can conclude that there is a range of the willingness to pay at which the presence of the public firm is irrelevant, namely in what we have called the kink equilibria. On the contrary, its presence modifies the nature and properties of market equilibrium when the market is very poor and the issue of the full market coverage becomes crucial, as is the case, for example, for social care, education, social housing, and water access.

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