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Consumer Choice under Certainty and Uncertainty

in Applied Econometrics

. by

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Abstract: The paper lists salient characteristics of the certainty theory of consumer choice and discusses the import of prominent empirical analyses of the theory. All of them reject the theory's empirical relevance which suggests that the theory is unfit to analyze consumer choice in an uncertain world. The paper presents an extension of the certainty theory to a theory about consumer choice under uncertainty in which consumer behavior has interesting new properties. I conclude with an empirical test of the empirical relevance of an uncertainty version of Stone's Linear Expenditure System. In the given empirical context Stone's System is empirically relevant.

Key words: Utility function, linear expenditure system, almost ideal demand system, income and substitution effects, expectations' effect.

JEL : A 12, B 23, B 41, C 01, C 18, C 30, C 45, C 51, C 52, D 12, D 41, D 59

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1. Introduction

Since 1950, the certainty theory of consumer choice has provided all the fundamental ideas underlying the study of consumer choice in applied econometrics. Econometricians use the theory to formulate their econometric models, to put restrictions on the values of the models' parameters, and to evaluate the theoretical and statistical adequacy of the respective models. Prominent examples of econometric models of consumer choice are J. R. N. Stone's (1954) *Linear Expenditure System*, H. Theil and A. P. Barten's (1965 and 1969) *Rotterdam Model*, L. R. D. Christensen, D. W. Jorgenson, and L. J. Lau's (1975) *Translog Model*, and A. S. Deaton and J. Muellbauer's (1980) *Almost Ideal Demand System.*

In Section 2, I describe the salient characteristics of the certainty theory of consumer choice; e.g., the homogeneity property of a Marshallian demand function and the symmetry and negative semi-definiteness property of a Hicksian demand function. In addition, I describe the way the mentioned empirical analyses use the theory, and discuss the import of their results; e.g., why the estimated Marshallian demand function is not homogeneous of degree zero, and why the estimated Hicksian demand function does not satisfy the theory's symmetry condition.

The failure of the certainty theory of consumer choice calls for a new theory. In Section 3, I describe the salient characteristics of such a theory – the uncertainty theory of consumer choice as introduced in Stigum 1969 and 1972. The new theory is a natural extension of the certainty theory. Yet, the properties of its Marshallian and Hicksian demand functions are very different from their properties in the certainty theory. For example, a consumer's Marshallian demand function need not be homogeneous of degree zero in prices and income. In addition, its value varies with changes in prices, not just because relative prices and the consumer's real income change, but also because the consumer's price expectations change. Similarly, a consumer's Hicksian demand function need not be homogeneous of degree zero in prices, and its matrix of partial derivatives with respect to prices need not be symmetric. In addition, the function varies with changes in prices, not just to adjust to changes in relative prices - like the Hicksian demand function in the certainty theory. The Hicksian demand function in the new theory varies with prices, also, because the consumer's real income and price expectations change with the change in prices. The properties of the uncertainty version of the consumer's cost function are, also, very different from its properties in the certainty theory. For example, the function in the new theory need not be concave and linearly homogeneous in prices.[1]

I conclude the paper in Section 4 by presenting an empirical analysis of an uncertainty version of Stone's Linear Expenditure System. In the given empirical

Note 1: In Chapters 10 and 30 of Stigum 1990, the theory of consumer choice under certainty and uncertainty is developed for consumers with continuous utility functions. To simplify my arguments in Sections 2 and 3 of the paper, I apply relevant details of the two theories assuming that the pertinent consumers' utility functions are twice differentiable. My references and examples show that there are consumers whose utility functions in each case satisfy my assumptions.

context, the uncertainty version of Stone's System is empirically relevant. The result is interesting, and the arguments that establish it involve a novel and intriguing interplay between theory and data in applied econometrics. This interplay adds new insight into the applied-econometric consequences of T. Haavelmo's idea of identifying the values of theoretical variables with the true values of pertinent data variables. It, also, helps me determine how a consumer in an empirically relevant uncertainty version of J.R.N. Stone's Linear Expenditure System reacts to changes in prices and net worth.

2. The Certainty Theory of Consumer Choice in Applied Econometrics

J. R. Hicks' *Value and Capital* (cf. Hicks, 1939, pp. 11- 52 and 305 − 314) and P. A. Samuelson's *Foundations of Economic Analysis* (cf. Samuelson, 1947, pp. 90 -124) provide a detailed account of the certainty theory of consumer choice. In its intended interpretation, the theory is about a consumer - an individual or a family living together - who faces a price vector, $p € R_{++}^n$, and is to choose a vector of commodities, $q € R_+^n$, that maximizes the value of his utility function, $U(\cdot): R_+^n \rightarrow R_+$, subject to his budget constraint, $\{q € R_+^n: pq \leq A\}$. ·Here $A € R_{++}$ denotes the consumer's net worth, and in R_+^n the utility function is continuous, strictly quasi-concave, and increasing. In addition, "in a wide region" (cf. Samuelson 1947, p. 29) $U(\cdot)$ is twice continuously differentiable, and the matrix, $\{\partial^2U(q)/\partial q_i\partial q_j\}$ is invertible, symmetric, and negative definite. To simplify my deliberations in this section, I take Paul Samuelson's wide region to be all of R_{++}^n.

2.1 The Marshallian demand function

In the certainty theory of consumer choice, the consumer's behavior can be described by a functional equation, $q = f(p, A)$, where $f(\cdot): R_{++}^{n+1} \rightarrow R_{+}^{n}$, is a differentiable function with many interesting properties. At each value of $(p, A) \in R_{++}^{n+1}$, $f(p, A)$ equals the vector in R_{+}^{n} at which $U(\cdot)$ attains its maximum value subject to the condition, $pq \leq A$. In addition, $f(\cdot)$ has three properties that provide a basis for statistical tests of the empirical relevance of the theory: For all $(p, A, \lambda) \in R_{++}^{n+2}$,

(1) *Adding up*: $pf(p, A) = A$ – i.e., the value of purchased commodities add up to the given net worth;

(2) *Homogeneity*: $f(\lambda p, \lambda A) = f(p,A)$ – i.e., $f(\cdot)$ is homogeneous of degree zero; and

(3) *Symmetry and negative semi-definiteness*: $\partial f_i(p, A)/\partial p_i + q_i \partial f_i(p,A)/\partial A < 0$, $i = 1,\ldots,n$, and the matrix, $\{\partial f_i(p, A)/\partial p_j + q_j \partial f_i(p,A)/\partial A\}$ with $i,j = 1,\ldots,n$, is symmetric and negative semi-definite.

Econometricians refer to $f(\cdot)$ as the consumer's *Marshallian demand function*.

The Marshallian demand function has a huge family of models. An interesting subfamily of models of $f(\cdot)$ is J. R. N. Stone's Linear Expenditure System. It contains 2n parameters, $b \in R_{+}^{n}$, and $c \in R_{+}^{n}$, where b is a vector of constants that sum to one, and c is a vector of commodities to which the consumer in some sense is committed (cf. Stone 1954, p. 512). The pair, (b, c), combines with q, p, and A to form Stone's econometric model,

$$q_i = c_i + b_i[(A - pc)/p_i],\ i = 1,\ldots,n;\ \sum_{1\leq i\leq n} b_i = 1, \text{ and } 0 \leq pc < A. \qquad (1)$$

The two conditions on b and c ensure that Stone's Linear Expenditure System makes sense, and that it has the adding-up and homogeneity properties of a Marshallian demand function. In addition, it is true that there are models of the consumer's utility function; e.g.,

$U(q) = \sum_{1\leq i\leq n} a_i \log(q_i - c_i)$, with $q \in R_+^n$, $a_i \in R_{++}$, and $0<c_i<q_i$, $i = 1,\ldots,n$, (2)

for which Stone's Linear Expenditure System is a Marshallian demand function. That goes to show that one can add conditions on the coefficients in (1) so that the system also satisfies the symmetry and negative semi-definite property of a Marshallian demand function. Those conditions are complicated. Stone spells them out for the symmetry property on pp. 513-514 in Stone 1954.

2.2 The Hicksian demand function

The Marshallian demand function itself is not symmetric. The symmetry and negative semi-definite property of $f(\cdot)$ claims that a compensated version of $f(\cdot)$ is symmetric. This compensated version, $h(\cdot)$, econometricians refer to as the *Hicksian demand function*. It satisfies the conditions: $h(\cdot): R_{++}^{n+1} \rightarrow R_+^n$, and at each value of (u, p) with $u \in R_{++}$ and $p \in R_{++}^n$, $h(u, p)$ equals the q at which pq attains its minimum value in the set, $\{q \in R_+^n: U(q) \geq u\}$. In addition, $h(\cdot)$ is differentiable in R_{++}^{n+1}, homogeneous of degree zero in p, and symmetric in the sense that it satisfies the conditions,

$$\partial h_i(u,p)/\partial p_j = \partial h_j(u,p)/\partial p_i \text{ for all, } i,j = 1,\ldots,n, \text{ and } (u,p) \in R_{++}^{n+1}.$$

Finally, for all $(u, p, A) \in R_{++}^{n+2}$ at which $u = U(f(p, A))$, it is a fact that

$$h(u, p) = f(p, A), \quad ph(u, p) = A,$$

$$\partial h_i u, p)/\partial p_j = \partial f_i(p, A)/\partial p_j + q_j \partial f_i(p,A)/\partial A, \; i,j = 1,\ldots,n, \text{ and}$$

the matrix, $\{ \partial h_i(u, p)/\partial p_j \}$, is symmetric and negative semi-definite. (3)

The equations in (3) explicate in what sense the Hicksian demand function, h(·), is a compensated version of the Marshallian demand function, f(·).

2.3 The cost function

The function, c(·): $R_{++}^{n+1} \rightarrow R_{++}$, which at each (u,p) € R_{++}^{n+1} satisfies the equation, c(u,p) = ph(u,p), econometricians refer to as the *consumer's cost function*. The function has many models and many interesting properties. For example, the following function is the cost function of a model of Stone's Linear Expenditure System with utility function as specified in equation (2): For all (u,p) € R_{++}^{n+1},

$$c(u,p) = pc + ug(a, c)\prod_{1 \leq j \leq n} p^{\beta j}, \quad (4)$$

where $\beta_j = (a_j/\Sigma_{1 \leq k \leq n} a_k)$, j =1,…,n., and where g(·): $R_{++}^{2n} \rightarrow R_{++}$ is a differentiable function of the parameters in equation (2). As to the properties of a consumer's cost function, it is differentiable, concave, and homogeneous of degree one in p, and satisfies the equations,

$$\partial c(u, p)/\partial p_i = h_i(u,p) \text{ for all } (u,p) € R_{++}^{n+1} \text{ and all } i = 1,\ldots,n. \quad (5)$$

In addition, it provides the theoretical foundation on which A. Deaton and J. Muellbauer construct their Almost Ideal Demand System.

The consumer's cost function, c(·), in the Almost Ideal Demand System satisfies the following equation: For all (u,p) € R_{++}^{n+1}.

$$\log c(u,p)=a_0 +\sum_{1\leq k\leq n} a_k \log p_k+(1/2)\sum_{1\leq k\leq n}\sum_{1\leq j\leq n} \gamma^*_{kj} \log p_k \log p_j+u\beta_0\Pi_{1\leq k\leq n} p_k^{\beta k} \quad (6)$$

When this function is a model of the cost function of a consumer, then for all (u, p) € R_{++}^{n+1}, $\partial c(u, p)/\partial p_i = q_i$, i = 1, …, n, where q = (q_1, …, q_n) is the vector in R_+^n at which pq attains its minimum value in the set, {q € R_+^n: U(q) ≥ u}. In addition, for all (u, p) € R_{++}^{n+1},

$$\partial \log c(u, p)/\partial \log p_i = w_i = a_i + \sum \gamma_{ij} \log p_j + \beta_i u \beta_0 \Pi_{1\leq k\leq n}\, p_k^{\beta k,}\ , \qquad (7)$$

where $w_i = p_i q_i / c(u, p)$, and $\gamma_{ij} = (1/2)(\gamma^*_{ij} + \gamma^*_{ji})$. Now, for a utility-maximizing consumer, his net worth equals the value of his cost function. Hence, if one substitutes A for c(u,p) in equation (6), solves the equation for u, and substitutes the solution in (6) for the u in equation (7), one ends up in (8) with the budget-share form of a Hicksian demand function in the Almost Ideal Demand System:

$$w_i = a_i + \sum_{1\leq j\leq n} \gamma_{ij} \log p_j + \beta_i \log(A/P),\ i = 1, \ldots, n;\ \text{and } (p,A) \text{ in } R_{++}^{n+1} \qquad (8)$$

where

$$\log P = a_0 + \sum_{1\leq k\leq n} a_k \log p_k + (1/2)\sum_{1\leq k\leq n}\sum_{1\leq j\leq n} \gamma^*_{kj} \log p_k \log p_j.$$

Deaton and Muellbauer think of the logP in (8) as the logarithm of an appropriately chosen price index. In addition, they observe that a model of the equations in (8) can be a Hicksian demand function of a consumer's utility function only if its parameters ensure that the model has the salient properties of h(·): The model has the adding-up property only if $\sum_{1\leq i\leq n} a_i = 1$, $\sum_{1\leq i\leq n} \gamma_{ij} = 0$, for all j = 1, …, n, and $\sum_{1\leq i\leq n} \beta_i = 0$; it has the homogeneity property only if $\sum_{1\leq j\leq n} \gamma_{ij} = 0$; it has the symmetry property only if $\gamma_{ij} = \gamma_{ji}$, for i,j = 1, ..., n; and it has the negative semi-definite property only if the matrix of functions, $E = \{e_{ij}\}$, where

$$e_{ij} = \gamma_{ij} + \beta_i \beta_j \log(A/P) - w_i \delta_{ij} + w_i w_j\ ,\ i,j = 1, \ldots, n. \qquad (9)$$

is negative semi-definite. Here δ_{ij} is the Kronecker delta that is 1 if i = j and 0 otherwise (cf. equation (14) and pertinent remarks on p. 316 in Deaton and Muellbauer 1980).

2.4 The empirical analyses

In an empirical analysis a user of Stone's linear expenditure system or a user of Deaton and Muellbauer's almost ideal demand system sets out to test the empirical relevance of the certainty theory of consumer choice. In doing that, he faces many serious problems two of which concern the references of his observations and the constraints on his model's parameters. For example, Stone analyses a system of six commodity groups. One of the groups contains meat, fish, dairy products, and fats. Another group contains household running expenses, non-durable household goods, and domestic services. Deaton and Muellbauer analyze a system of eight groups. One of these groups is called foods. Another group is called transport and communications services. How is one to define q and p for such commodity groups? The theory does not specify what the components of q and the components of p measure. Consequently, Stone and Deaton and Muellbauer – in the spirit of Trygve Haavelmo's 1944 Treatise (cf. Haavelmo 1944, pp. 6-8) – can identify the values of their theoretical variables with the true values of the prices and quantities that their British data provide, and assume that they have accurate observations of the true values of the pertinent prices and quantities. Then the given British data end up constituting the context in which the authors test the empirical relevance of the certainty theory of consumer choice.

The constraints on their models' parameters pose a problem for both Stone and Deaton and Muellbauer. Should the conditions on b and c in equation (1) and the conditions on a_i, γ_{ij}, β_i, and e_{ij} in equations (8) and (9) be integral parts of the respective econometric models? Stone imposed the restrictions when he estimated the values of b and c. Deaton and Muellbauer tried both ways. Whether their choices made a difference as to their tests of the empirical relevance of the certainty theory of consumer choice is uncertain. Stone's analysis provides no answer, and Deaton and Muellbauer in both analyses reject the theory. The values of the estimated parameters are such that the resulting models of the equations in (8) do not satisfy the homogeneity and symmetry property of a Hicksian demand function.

Deaton and Muellbauer are not the only ones who have rejected the certainty theory of consumer choice. For example, Barten used the Rotterdam model to analyze Dutch consumers' choice of nine commodity groups during the period 1922-1939 and 1949-1962. He found that "the homogeneity condition has to be rejected" and that "the composite hypothesis of the condition of homogeneity and the symmetry condition must also be rejected" (Barten 1969, p.78). Deaton used the Rotterdam model to analyze British consumers' choice of nine commodity groups during the period 1900-1914, 1921-1039, and 1953-1970. He ended up rejecting both the homogeneity condition and the symmetry condition. Finally, Christensen, Jorgenson, and Lau used their own Translog model to analyze U.S. consumers' choice of three commodity groups during the period 1929-1972. They conclude "that the theory of demand is inconsistent with the evidence." In all these cases the authors estimated both unrestricted and restricted versions of their econometric models and based their conclusion as to the empirical relevance of the restricted versions of their econometric models on a large-sample likelihood ratio test.

2.5 Concluding remarks

The authors of the mentioned articles suggest that measurement errors of their variables, too few explanatory variables, and misspecification of the dynamics of the data generating process are possible causes of the failure of the certainty theory of consumer choice in their empirical analyses. I believe that the reason for the theory's failure is much more fundamental. Consumers live in an uncertain world, and the certainty theory of consumer choice is unfit to analyze the problems of consumer choice under uncertainty.

In Section 3, I will discuss an uncertainty theory of consumer choice that was introduced in Stigum 1969, 1972, and 1990. This uncertainty theory is a natural extension of the certainty theory to a theory of consumer choice under uncertainty. Even so, in the new theory the Marshallian demand function need not satisfy the homogeneity and symmetry conditions stated above. In addition, the definitions and the imports of both the Hicksian demand function and the cost function in the new theory are controversial.

3. The Uncertainty Theory of Consumer Choice in Applied Econometrics

It is a fact that Bernt Stigum's uncertainty theory of consumer choice is a natural extension of the certainty theory. To see why, consider the following subfamily of models of the certainty theory in which n = 2 and U(·) is an integral; i.e., in which there exists a twice differentiable function, $V(\cdot):R_+\rightarrow R_+$, and a cumulative probability distribution, $F(\cdot):R_+\rightarrow[0, 1]$, that satisfy the conditions:

$V'(\cdot) > 0$, $V''(\cdot) < 0$, $\int_{(0,\infty)} dF(r) = 1$, and $U(q_1, q_2) = \int_{(0,\infty)} V(q_1 + q_2 r) dF(r)$. In addition, assume that $p = (1, p_2)$, think of q_1 and q_2, respectively, as so many units of the unit of account and as so many units of a risky asset, take p_2q_2 to record the value of the consumer's investment in q_2, and assume that q_2r measures the value of q_2 the "next period". With this subfamily of models of the certainty theory one can develop all of Kenneth Arrow and John Pratt's theory of choice among safe and risky assets (see Arrow 1965 and Pratt 1964). Chapters 10 and 12 in Stigum 1990 show how.

Arrow and Pratt's theory is a very interesting theory of choice under uncertainty. Yet, for the purposes of this paper, it has one serious defect. The probability distribution of the next period prices of q, (1, r), is independent of the current price of q. Suppose that the distribution of (1,r) depends on the value of (p_1, p_2). Then the consumer' utility function will receive two more arguments as witnessed in the equations in (10):

$$U(p_1,p_2,q_1,q_2) = \int_{(0,\infty)} V(q_1 + q_2 r) dF(r \mid p_1,p_2), \text{ and } \int_{(0,\infty)} dF(r \mid p_1,p_2) = 1. \qquad (10)$$

The change looks innocuous, but it is not. Arrow and Pratt's theory is not valid with a utility function and a conditional probability distribution like the one in (10).

3.1 Consumer choice under uncertainty

The utility function in (10) is like a prototype of the utility function in the new theory of consumer choice under uncertainty. Think of a consumer who has a finite planning horizon – say T periods - and suppose that he orders T commodity vectors, $q_i \in R_+^n$, i = 1, …, T, and a period-T risky asset, M_T, according to the values of an increasing, strictly concave, and a.e. twice continuously differentiable function, $V(\cdot):R_+^{nT+1} \rightarrow R_+$. In addition, suppose that he has observed the values of the first-period prices, $(p_1,p_{M1}) \in R_{++}^{n+1}$, and that he - conditioned on the observed value of (p_1,p_{M1}) - has a well-defined subjective probability distribution of the vector, $((p_2,p_{M2})\ldots, (p_T, p_{MT}))$, where $(p_i,p_{Mi}) \in R_{++}^{n+1}$, i = 2, …, T. Finally, suppose that he chooses his first-period commodity vector, q_1, his first-period investment in risky assets, M_1, and his plans for purchases of future commodities and for investments in risky assets so that he maximizes the expected value of $V(q_1, \ldots, q_T, M_T)$ subject to the budget constraints that he faces in each period. Under reasonable conditions on the consumer's subjective probability distribution, one can show that there exists a function,

$$U(\cdot):R_{++}^{n+1} \times R_+^{n+1} \rightarrow R_+, \qquad (11)$$

such that the consumer in the first period – conditional on the observed values of p_1 and p_{M1} – chooses a q_1 and an M_1 that will

$$\text{maximize the value of } U(p_1,p_{M1},q_1,M_1) \qquad (12)$$

subject to his first-period budget constraint,

$$\{(q_1,M_1) \in R_+^{n+1}:p_1q_1+p_{M1}M_1 \leq A\}. \qquad (13)$$

Here A is his first-period initial net worth, and $U(\cdot)$ is continuous in (p_1,p_{M1},q_1,M_1), and increasing and strictly concave in (q_1, M_1). For the purposes of this paper, I assume that $U(\cdot)$ is twice differentiable in (p_1,p_{M1},q_1,M_1).

Here an example may be of help. The example describes a two-period model of the theory of consumer choice under uncertainty that I sketched above. In this context the model is interesting because it constitutes a two-period uncertainty version of Stone's Linear Expenditure System.

Example 1 In this example, $T = 2$, $n=2$, $q_1 = (x_1, x_2)$, $q_2 = (y_1, y_2)$, and $M_i = \mu_i$, $i = 1,2$. In addition, $P_1 = (p_{11}, p_{12}, p_{13})$, $P_2 = (p_{21}, p_{22}, p_{23})$, $P_i \in R_{++}^3$, $i = 1,2$, and the coefficients that appear in the consumer's utility function, a, b, c, d, e, α, β, δ, γ, are all positive. Finally, the consumer's utility function is as follows:

$$V(q_1,q_2,M_2)=\alpha\log(x_1-\delta)+\beta\log(x_2-\gamma)+[a\log(y_1-d)+b\log(y_2-e)+ c\log\mu_2]$$

The two budget constraints are

$$p_{11}x_1 + p_{12}x_2 + p_{13}\mu_1 \leq A, \text{ and } p_{21}y_1 + p_{22}y_2 + p_{23}\mu_2 \leq \mu_1.$$

For a given value of P_2, the maximum value of the consumer's second-period utility is given in equation (14):

$$B(P_2) + (a+b+c)\log(\mu_1 - p_{21}d - p_{22}e), \text{ with } \mu_1 - p_{21}d - p_{22}e > 0, \qquad (14)$$

where

$$B(P_2)=a\log((a/p_{21})/(a+b+c))+b\log((b/p_{22})/(a+b+c))+c\log((c/p_{23})/(a+b+c))$$

The components of P_2 in (14) are random variables whose probability distribution depends on the observed value of P_1. Consequently, the $U(\cdot)$ that the consumer is to maximize in the first period is as follows,

$$U(p_1,p_{M1},q_1,M_1) = \alpha\log(x_1 - \delta) + \beta\log(x_2 - \gamma) +$$
$$E[B(P_2) + (a+b+c)\log(\mu_1 - p_{21}d - p_{22}e) \mid P_1], \qquad (15)$$

where $E[(\cdot) \mid P_1]$ denotes the conditional expectation of $(\cdot)$ given the observed value of P_1.

3.2 The Marshallian demand function

The Marshallian demand function in the uncertainty theory of consumer choice is a function, $F(\cdot):R_{++}^{n+2} \to R_+^n$, that for each $(p_1, p_{M1},A) \in R_{++}^{n+2}$ records the value of the vector, $(q_1, M_1) \in R_+^{n+1}$, at which $U(p_1,p_{M1}, \cdot)$, attains its maximum value in the set, $\{(q_1,M_1) € R_+^{n+1}: p_1q_1 + p_{M1}M_1 \leq A\}$.[2]

The Marshallian demand function has many models, and the properties of the models depend on the roles one assigns to the components of (p_1, p_{M1}) in $U(p_1,p_{M1}, \cdot)$. To each subfamily of models of $U(p_1,p_{M1}, \cdot)$ corresponds a subfamily of models of the Marshallian demand function. In one subfamily of models of $U(p_1,p_{M1}, \cdot)$, the pair, (p_1, p_{M1}) may assume an arbitrary fixed value of current period prices that carries no information about future prices. That will be the case when the probability distribution of future prices is independent of the values of current-period prices. In another subfamily, $U(p_1,p_{M1}, \cdot)$ will vary with (p_1, p_{M1}), but its indifference surfaces will not since they in the given subfamily are independent of (p_1,p_{M1}). I give an example of such a case in Example 2 below. There may, also, be subfamilies of models of $U(p_1,p_{M1}, \cdot)$ in which its (p_1, p_{M1}) arguments assume different values when the observed (p_1,p_{M1}) vary over a subset of R_{++}^{n+1} - say B - and assume an arbitrary

Note 2: In his 1968 article on consumer behavior, Peter J. Kalman introduced prices in the consumer's utility function. His aim was to extend the certainty theory of consumer choice so that it would allow for two possibilities that seemed important to him. One was that a consumer may judge the quality of a commodity by its price. The other was that a consumer's demand function need not be linearly homogeneous in prices and money income. Kalman's references for the first possibility were T. Scitovsky 1945, L.Thurstone 1931, and T. Veblen 1912. His reference for the second possibility was J. Marschak 1943.

constant value when the observed values of (p_1, p_{M1}) belong to the complement of B. The subfamily of Marshallian demand functions that the utility function in (15) determines may be like that.

Example 2 In this example, T = 2, n=2, $q_1 = (x_1, x_2)$, $q_2 = (y_1, y_2)$, and $M_i = \mu_i$, i = 1,2. In addition, $P_1 = (p_{11}, p_{12}, p_{13})$, $P_2 = (p_{21}, p_{22}, p_{23})$, P_i € R_{++}^3, i = 1,2. Finally, the consumer's utility function is as follows:

$$V(q_1,q_2,M_2) = x_1 \cdot x_2 \cdot y_1^{1/3} \cdot y_2^{1/3} \cdot \mu_2^{1/3}. \qquad (16)$$

The two budget constraints are

$$p_{11}x_1 + p_{12}x_2 + p_{13}\mu_1 \leq A, \text{ and } p_{21}y_1 + p_{22}y_2 + p_{23}\mu_2 \leq \mu_1.$$

For a given value of P_2, the maximum value of the $V(q_1,\cdot)$ in (16) subject to the second-period budget constraint is given in equation (17):

$$V(q_1, (\mu_1/3p_{21})^{1/3}, (\mu_1/3p_{22})^{1/3}, (\mu_1/3p_{23})^{1/3}) = (1/27(p_{21} \cdot p_{22} \cdot p_{23}))^{1/3} x_1 \cdot x_2 \cdot \mu_1. \quad (17)$$

The components of P_2 in (17) are random variables whose probability distribution depends on the observed value of P_1. Consequently, the U(·) that the consumer is to maximize in the first period is as follows,

$$U(p_1,p_{M1},q_1,M_1) = E[(1/27(p_{21} \cdot p_{22} \cdot p_{23}))^{1/3} \mid P_1] \cdot x_1 \cdot x_2 \cdot \mu_1, \qquad (18)$$

where $E[(\cdot) \mid P_1]$ denotes the conditional expectation of (·) given the observed value of P_1.

The family of utility functions that the utility function in (18) determines is a subfamily of models in which the members share one and the same family of indifference surfaces. Moreover, the Marshallian demand functions are as follows: $x_1 = A/3p_{11}$: $x_2 = A/3p_{12}$; and $\mu = A/3p_{13}$.

One obtains a Marshallian demand function by solving the necessary conditions for a constrained maximum of a utility function, $U(p_1,p_{M1}, \cdot)$:

$$\partial U(p_1,p_{M1}, q_1,M_1)/\partial q_{1i} = \lambda p_{1i},\ i = 1,\ldots, n;$$

$$\partial U(p_1,p_{M1}, q_1,M_1)/\partial M_1 = \lambda p_{M1};\ \text{and}$$

$$p_1q_1+p_{M1}M_1 = A, \qquad (19)$$

where λ is the Lagrange multiplier. The properties of $U(p_1,p_{M1}, \cdot)$ ensure that the Marshallian demand function at (p_1,p_{M1},A), $F(p_1,p_{M1},A)$, is well defined. In this paper I assume that $F(\cdot)$ is differentiable.

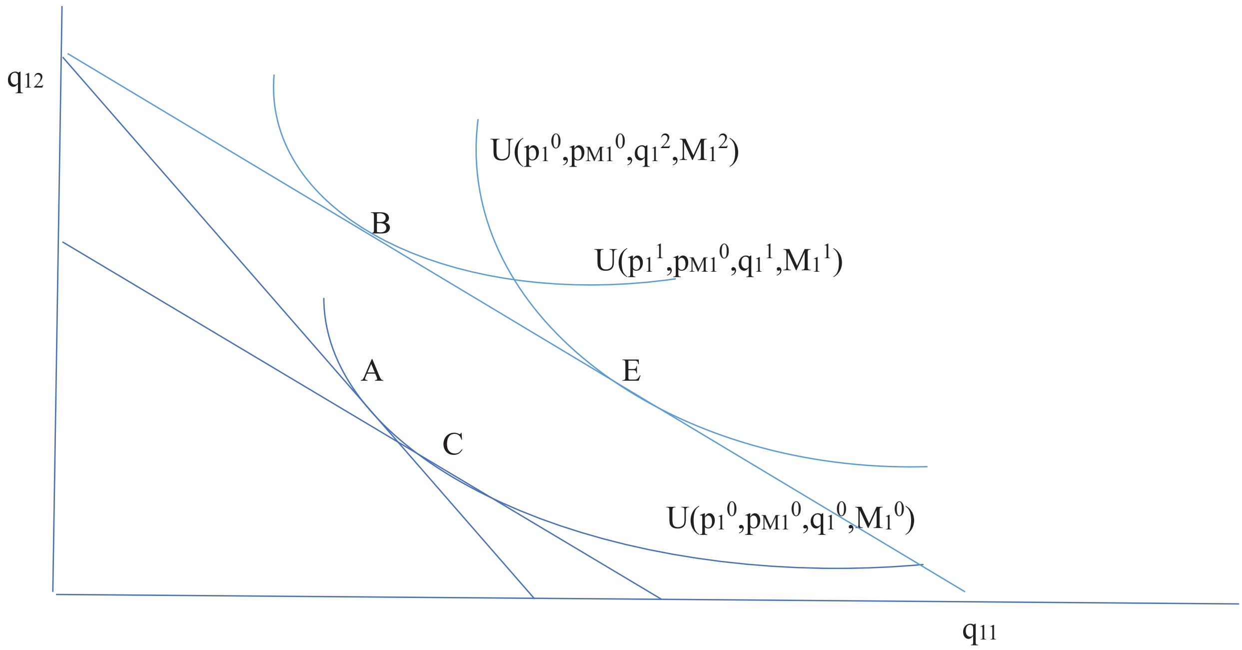
The uncertainty version of $f(\cdot)$ satisfies the adding-up condition of the certainty version. Whether it satisfies the homogeneity condition, depends both on the consumer's price expectations and on his utility function. For example, it satisfies the homogeneity condition if the consumer's probability distribution of future prices is independent of current-period prices, or if his utility function is like the utility function in Example 2. However, there are models of the uncertainty version of Stone's Linear Expenditure System that do not satisfy the homogeneity condition. Whether the uncertainty version of the Marshallian demand function satisfies the symmetry condition of the certainty version, is a question that I will bring up later in my discussion of the uncertainty version of the Hicksian demand function.

3.3 The expectations effect

The Marshallian demand function in the present theory has properties that the certainty theory's Marshallian demand function does not have. The idea behind Figure 1 is to exhibit one of these properties – the expectations' effect on consumer choice of a change in current-period prices. In the figure, A and B, respectively, are names of the vectors, (q_1^0,M_1^0) and (q_1^1,M_1^1), at which $U(p_1^0,p_{M1}^0,\cdot)$ and $U(p_1^1,p_{M1}^0,\cdot)$ attain their maximum values in the sets, $\{q_1,M_1) € R_+^{n+1}: p_1^0q_1 + p_{M1}^0M_1 \leq A\}$ and $\{(q_1,M_1) € R_+^{n+1}: p_1^1q_1 + p_{M1}^0M_1 \leq A\}$.

Figure 1

The Effect on (q_{11},q_{12}) of a Change in the Value of p_{11}



In addition, $U(p_1^0,p_{M1}^0,q_1^0,M_1^0)$, $U(p_1^1,p_{M1}^0,q_1^1,M_1^1)$, and $U(p_1^1,p_{M1}^0,q_1^2,M_1^2)$ are, respectively, names of the indifference curves going through A, B, and E. The figure illustrates how one gets from A to B. Keeping the current prices, p_1^0,p_{M1}^0, fixed in $U(\cdot)$, the standard substitution and income effects of a change in prices from (p_1^0,p_{M1}^0) to (p_1^1,p_{M1}^0), moves the equilibrium in A, first, to the equilibrium in C, and then, to the equilibrium in E. When the values of the prices in $U(\cdot)$ change as well, the expectations effect moves the equilibrium in E to the equilibrium in B.

Before I delineate the mathematical arguments that underlie Figure I, a comment about the generality of the situation that the figure portrays is necessary. Certainly, when the probability distribution of future prices is independent of the observed current-period prices, the equilibrium in B coincides with the equilibrium in E. The same is the case when the consumer's utility function is like the one in Example 2. This is so, because, then, the price change does not affect the consumer's indifference curves. These examples notwithstanding, I believe that the idea behind Figure 1 is valid for most applied-econometrics analyses in our uncertain world. Therefore, in my mathematical deliberations I will argue as if the exceptions do not exist.

To derive the mathematical arguments that underlie Figure 1, it is necessary to establish salient characteristics of the derivatives of $F(\cdot)$ with respect to p_1, p_{M1}, and A. For that purpose, I let q_{1n+1} be M_1, and p_{1n+1} be p_{M1}, and I let $U_{x,y} = \partial^2 U(p_1,p_{M1}, \cdot)/\partial x \partial y$, where x and y vary over q_1s, M_1, p_1s , and p_{M1}. Then, it follows by standard arguments from the equations in (19) - the necessary conditions for a constrained maximum of $U(p_1,p_{M1},\cdot)$ - that

$$\partial F_i(p_1,p_{M1},A)/\partial p_{1j} =$$

$$D^{-1}[\lambda\, D_{ji} - q_{1j}D_{(n+2)i}] - D^{-1}\sum_{1\le k\le n+} D_{ki}\partial^2 U(p_1,p_{M1},q_1,M_1)/\partial q_{1k}\partial p_{1j},\ i,j = 1,\ldots, n+1,$$

$$\partial F_i(p_1,p_{M1},A)/\partial A = D^{-1} D_{(n+2)i}\ ,\ i = 1,\ldots, n+1. \qquad (20)$$

In these equations, λ is the Lagrange multiplier,

$$D = \text{determinant of} \begin{pmatrix} U_{q11,q11} & U_{q11,q12} & \ldots\ \ldots & U_{q11,q1n} & U_{q11,M1} & -p_{11} \\ \ldots & \ldots & & \ldots & \ldots & \\ U_{q1n,q11} & U_{q1n,q12} & \ldots\ \ldots & U_{q1n,q1n} & U_{q1n,M1} & -p_{1n} \\ U_{q1n+1,q11} & U_{q1n+1,q12} & \ldots\ \ldots & U_{q1n+1,q1n} & U_{q11,M1} & -p_{1M1} \\ p_{11} & p_{12} & \ldots\ \ldots & p_{1n} & p_{1M1} & 0 \end{pmatrix}$$

and D_{ij} is the cofactor of the ij^{th} element in the matrix of D.

From the equations in (20) it follows that a change in a price - say p_{1j} - has three effects on each of the components of $F(p_1,p_{M1},A)$. The effects on $F_i(p_1,p_{M1},A)$ are

I. a Substitution Effect: $\lambda D^{-1}D_{ji}$;

II. an Income Effect: $-q_{1j}D^{-1}D_{(n+2)i}$; and

III. an Expectations Effect: $-D^{-1}\sum_{1\leq k\leq n+1} D_{ki}\partial^2U(\cdot)/\partial q_{1k}\partial p_{1j}$.

Since the entries in D concerning prices are fixed initial values of the p_{1j}, j = 1,…,n and p_{M1}, the substitution and income effects are like the substitution and income effects in the certainty theory of consumer choice. The substitution effect moves the equilibrium in A in Figure 1 to C. The income effect moves the equilibrium in C in Figure 1 to E. When the prices in the utility function change, the indifference curves of U(·) change in accord with the resulting change in the consumer's expectations. Such a change moves the equilibrium in E in Figure 1 to the equilibrium in B in accord with the mathematical description of the expectations' effect.

The next example will show what the expectations effect looks like in the uncertainty version of Stone's Linear Expenditure System.

Example 3 In the uncertainty version of Stone's Linear Expenditure System, the utility function to be maximized in the first period is

$$U(p_1,p_{M1},q_1,M_1) = \alpha\log(x_1-\delta) + \beta\log(x_2-\gamma) + E\{B(P_2) + (a+b+c)\log(\mu_1 - p_{21}d - p_{22}e) \mid P_1\}.$$

The necessary conditions for a maximum of this function subject to the budget constraint, $p_{11}x_1 + p_{12}x_2 + p_{13}\mu_1 \leq A$, are as follows:

$$\alpha/(x_1 - \delta) = \lambda p_{11};\ \beta/(x_2 - \gamma) = \lambda p_{12};$$

$$E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-1} \mid P_1] = \lambda p_{13};\ \text{and}\ p_{11}x_1 + p_{12}x_2 + p_{13}\mu_1 = A,$$

where λ is the Lagrange multiplier.

One can use the necessary conditions for a maximum to determine how the consumer's demand for x_1 changes when p_{11} changes. By standard

$$\begin{pmatrix} -\alpha/(x_1-\delta)^2 & 0 & 0 & -p_{11} \\ 0 & -\beta/(x_2-\gamma)^2 & 0 & -p_{12} \\ 0 & 0 & -E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-2} \mid P_1] & -p_{13} \\ p_{11} & p_{12} & p_{13} & 0 \end{pmatrix},$$

arguments - with D being the determinant of the matrix above, with Dij, i,j = 1,…,4 being the cofactor of the ijth element of this matrix, and with $\partial/\partial p_{11}E[\cdot \mid P_1]$ being the partial derivative of $E[\cdot \mid P_1]$ with respect to the p_{11}- the first component of P_1 - one finds in this case that

$$\partial F_1(p_{11},p_{12}, p_{13},A)/\partial p_{11} =$$

$$D^{-1}[\lambda D_{11} - x_1D_{41}] - D^{-1}\partial/\partial p_{11}E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-1} \mid P_1]D_{31}, \quad (21)$$

But if that is so, then in the uncertainty version of Stones econometric model the effect on the demand for x_1 when its price changes, can be analyzed in terms of three effects,

a substitution effect , $\lambda D^{-1}D_{11}$,

an income effect, $-x_1D^{-1}D_{41}$, and

an expectations effect, $-\partial/\partial p_{11}E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-1} \mid P_1]D^{-1}D_{31}$.

The expectations effect is missing in Stone's empirical analysis

3.4 The Hicksian demand function

There are many ways to define the uncertainty version of a Hicksian demand function. I will choose a way that goes with the uncertainty version of the Marshallian demand function that I described above.

The Hicksian demand function in the uncertainty theory of consumer choice is a function,

$$H(\cdot): R_{+} \times R_{++}^{n+1} \rightarrow R_{+}^{n+1}, \qquad (22)$$

that for each u € R_{+} and (p_1, p_{M1}) € R_{++}^{n+1} with (u,p_1,p_{M1}) € $R_{+} \times R_{++}^{n+1}$, records the value of the vector, (q_1, M_1) € R_{+}^{n+1}, at which the function, $p_1q_1 + p_{M1}M_1$, attains its minimum value in the set,

$$\{(q_1,M_1) € R_{+}^{n+1}: U(p_1, p_{M1}, q_1, M_1) \geq u\}. \qquad (23)$$

One obtains the Hicksian demand function by solving the necessary conditions for a constrained minimum of $p_1q_1 + p_{M1}M_1$ in (23). The necessary conditions for a minimum are as follows:

$$\lambda^{*}\partial U(p_1, p_{M1}, q_1, M_1)/\partial q_{1i} = p_{1i},\ i = 1, \ldots, n+1, \text{ and}$$

$$U(p_1, p_{M1}, q_1, M_1) = u, \qquad (24)$$

where λ^{*} is the Lagrange multiplier. Due to the properties of U(p_1, p_{M1},·), H(·) is well-defined at any (u, p_1,p_{M1}) € $R_{+} \times R_{++}^{n+1}$. It is continuous in all its arguments and, whenever u = U(p_1,p_{M1}, F(p_1, p_{M1},A)), it satisfies the conditions,

$$H(u,p_1,p_{M1}) = F(p_1, p_{M1},A), \text{ and}$$

$$(p_1, p_{M1})\ H(u,p_1,p_{M1}) = A, \qquad (25)$$

For the purposes of this paper, I will assume that H(·) is differentiable in all its arguments.

The uncertainty version of the Hicksian demand function differs in interesting ways from the certainty version. To see how, it is necessary to study properties of the derivatives of $H(\cdot)$. For that purpose, let D^* and D^*_{ji}, j,i = 1,…,n+1, be the

$$\begin{pmatrix} \lambda^* U_{q11,q11} & \lambda^* U_{q11,q12} & \cdots & \lambda^* U_{q11,q1n+1} & \partial U(\cdot)/\partial q_{11} \\ \cdots & \cdots & & \cdots & \cdots \\ \lambda^* U_{q1n+1,q11} & \lambda^* U_{q1n+1,q12} & \cdots & \lambda^* U_{q1n+1,q1n+1} & \partial U(\cdot)/\partial q_{1n+1} \\ \partial U(\cdot)/\partial q_{11} & \partial U(\cdot)/\partial q_{12} & \cdots & \partial U(\cdot)/\partial q_{1n+1} & 0 \end{pmatrix}$$

determinant and ji[th] cofactor of the matrix above. Then it follows by standard arguments from the constrained necessary minimum conditions in (24) that

$$\partial H_i(u,p_1,p_{M1})/\partial p_{1j} =$$

$$D^{*-1}[D^*_{ji} - \partial U(\cdot)/\partial p_{1j}\, D^*_{(n+2)i}] - D^{*-1}\textstyle\sum_{1\le k\le n+1} D^*_{ki}\lambda^*\partial^2 U(\cdot)/\partial q_{1k}\partial p_{1j},\ i,j = 1,\ldots,n+1.$$

$$\partial H_i(u,p_1,p_{M1})/\partial u = D^{*-1}\, D^*_{(n+2)i}\,. \qquad (26)$$

It is interesting to see how these equations differ from the equations in (20) that describe the derivatives of the Marshallian demand functions. If one multiplies the last column and last row of D^* by λ^*, the resulting determinant and ji[th] cofactors of the matrix of D^* equal, respectively,

$$\lambda^{*2}D^*,\ \ \lambda^{*2}D^*_{ji}, \text{ and } \lambda^* D^*_{(n+2)i} \ \text{ for } i,j = 1, \ldots, n+1.$$

In addition, whenever $u = U(p_1,p_{M1}, F(p_1, p_{M1},A))$,

$$\lambda^{*-n}D^* = -\lambda^2 D,\ \lambda^{*-(n-1)}D^*_{jj} = -\lambda^2 D_{jj},\ i,j = 1, \ldots, n+1, \text{ and}$$

$$\lambda^{*-(n-1)}D^*_{(n+2)i} = -\lambda D_{(n+2)i},\ i = 1,\ldots,n+1.$$

Consequently, for j,i = 1, …, n+1, and $u = U(p_1,p_{M1}, F(p_1, p_{M1},A))$,

$$D^*_{ji}/D^* = \lambda D_{ji}/D, \text{ and } D^*_{(n+2)i}/D^* = D_{(n+2)i}/D. \qquad (27)$$

But if that is so, then in terms of D and D_{ij} the derivatives of $H(\cdot)$ become

$$\partial H_i(u,p_1,p_{M1})/\partial p_{1j} =$$

$$D^{-1}[\lambda D_{ji} - \partial U(\cdot)/\partial p_{1j} D_{(n+2)i}] - D^{-1}\sum_{1\leq k\leq n+1} D_{ki}\partial^2 U(\cdot)/\partial q_{1k}\partial p_{1j}; \text{ and}$$

$$\partial H_i(u,p_1,p_{M1})/\partial u = D^{-1}D_{(n+2)i}, \qquad (28)$$

It follows from the equations in (28) that a change in a current-period price, say p_{1j}, has three effects on $H_i(\cdot)$, for i,j = 1, …, n+1,

a substitution effect, $D^{-1}\lambda D_{ji}$;

an income effect, $-D^{-1}\partial U(\cdot)/\partial p_{1j} D_{(n+2)i}$; and

an expectations' effect, $-D^{-1}\sum_{1\leq k\leq n+1} D_{ki}\partial^2 U(\cdot)/\partial q_{1k}\partial p_{1j}$. (29)

Whenever $u = U(p_1,p_{M1}, F(p_1, p_{M1},A))$, the Hicksian substitution effect is identical with the Marshallian substitution effect, and the Hicksian expectations' effect equals the Marshallian expectations' effect. Moreover, the Hicksian income effect differs from the Marshallian income effect both because $\partial U(\cdot)/\partial p_{1j}$ need not equal q_{ij}, and because the interpretation of the two effects differs. The Marshallian income effect is a measure of the change in the consumer's current-period real income due to a change in a current-period price. The Hicksian income effect is a measure of the change in the current-period real income due to a change in the consumer's expectations about the behavior of future prices.

It is, also interesting in this context to observe that, for all $(u, p_1, p_{M1}) \in R_{++}^{n+2}$ at which $u = U(p_1,p_{M1}, F(p_1, p_{M1},A))$, if one compensates $H(\cdot)$ for both an income effect and an expectations effect, $H(\cdot)$ becomes a compensated Marshallian demand function. To wit, for i,j = 1,…, n+1,

$$H(u, p_1,p_{M1}) = F(p_1, p_{M1},A)),\ D^{*-1}D^*_{ji} = D^{-1}\lambda D_{ji},$$

$$D^{*-1}D^*_{ji} = \partial H_i(u,p_1,p_{M1}))/\partial p_{1j} + \partial U(\cdot)/\partial p_{1j}\partial H_i(u,p_1,p_{M1})/\partial u +$$

$$D^{-1}\sum_{1\leq k\leq n+1} D_{ki}\partial^2 U(\cdot)/\partial q_{1k}\,\partial p_{1j}. \text{ and}$$

$$D^{-1}\lambda D_{ji} = \partial F_i(p_1,p_{M1},A)/\partial p_j + q_{1j}\cdot\partial F_i(p_1,p_{M1},A)/\partial A +$$

$$D^{-1}\sum_{1\le k\le n+1} D_{ki}\partial^2 U(\cdot)/\partial q_{1k}\partial p_{1j}.^3 \qquad (30)$$

The Hicksian demand function, H(·), need not be homogeneous of degree zero in prices. In addition, the matrix, $\{\ \partial H_i(u, U(p_1,p_{M1}), p_1, p_{M1})/\partial p_{1j}\ \}$ need not be symmetric and negative semi- definite. Only when the Hicksian demand function is compensated for both the income and expectations' effects, is it symmetric and negative semi-definite. The next example establishes that.

Example 4. Consider the uncertainty version of Stone's Linear Expenditure System with the utility function to be maximized,

$$U(p_1,p_{M1},q_1,M_1) = \alpha\log(x_1 - \delta) + \beta\log(x_2 - \gamma) +$$

$$E\{B(P_2) + (a+b+c)\log(\mu_1 - p_{21}d - p_{22}e)\ |\ P_1\},$$

Note 3: In this context it is interesting that the second equation in (28) is almost equal to Kalman's equation (2.4) which he established for the partial derivative, $\partial q_i/\ \partial p_h\,|\,u$ constant. In Kalman's derivative, u denotes the value of the Kalman-consumer's utility function and q_i and p_h, respectively, denote the i^{th} commodity and h^{th} price in Kalman's model. With my notation, Kalman's equation (2,4) equals,

$$D^{-1}\lambda D_{ji} - D^{-1}\sum_{1\le k\le n+1} D_{ki}\partial^2 U(\cdot)/\partial q_{1k}\partial p_{1j} - \lambda^{-1}\partial U(\cdot)/\partial p_{1i}D^{-1}D_{(n+2)i}.$$

To Kalman, his equation (2.4) records the substitution effect on q_i of a change in p_h (cf. Kalman's equations 2.4 and 2.5 and his comments on pp. 501 and 502). That is very different from the interpretation I give in (29) to the equations in (28).

In this case with i,j = 1, 2, 3, the matrix

$$\{\partial H_i(u,p_1,p_{M1})/\partial p_{1j}\} =$$

$$\{\lambda D_{ji}/D - \partial U(\cdot)/\partial p_{1j}\, D_{4i}/D - \partial/\partial p_{11} E[(a+b+c)\,(\mu_1 - p_{21}d - p_{22}e)^{-1} \,|\, P_1\,] D_{31}/D\,\}.$$

Since,

$$D_{21} = (p_{11}\cdot p_{12}\cdot E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-2} \,|\, P_1\,],$$

$$D_{12} = (p_{11}\cdot p_{12}\cdot E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-2} \,|\, P_1\,]),$$

$$D_{41} = (p_{11}\cdot\beta\cdot E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-2} \,|\, P_1\,])/(x_2 - \gamma)^2, \text{ and}$$

$$D_{42} = (\alpha p_{12}\cdot E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-2} \,|\, P_1\,])/(x_1 - \delta)^2,$$

it is evident that $\partial H_2(u,p_1,p_{M1})/\partial p_{11}$ need not equal $\partial H_1(u,p_1,p_{M1})/\partial p_{12}$, and hence that the given matrix need not be symmetric. However, when compensated for both the income and expectations effect, the given Hicksian demand function is symmetric.

3.5 The cost function under uncertainty

The consumer's cost function in the theory of consumer choice under uncertainty, $C(\cdot)$, records the minimum value of $p_1q_1 + p_{M1}M_1$ in the set, $\{(q_1,M_1) \in R_+^{n+1}: U(p_1,p_{M1},q_1,M_1) \geq u\}$. Consequently,

$$C(\cdot): R_+ \times R_{++}^{n+1} \rightarrow R_+^{n+1}, \text{ and}$$

$$C(u,p_1, p_{M1}) = \textstyle\sum_{1\leq i\leq n+1} p_{1i}H_i(u,p_1, p_{M1}). \qquad (31)$$

The present cost function is an increasing function of u. However, since $U(p_1,p_{M1},\cdot)$ may change with the change in prices, $C(u,\cdot)$ need not be a non-decreasing and concave function of (p_1, p_{M1}). Moreover, since $H(u,\cdot)$ need not be homogeneous of degree zero, $C(u,\cdot)$ need not be homogeneous of degree one. Finally, the effect on $C(\cdot)$ of a change in a price, say $\partial C(u,p_1, p_{M1})/\partial p_{1i}$, need not equal $H_i(u,p_1,p_{M1})$.

To see why, refer to (28) and observe that

$$\partial C(u, p_1, p_{M1})/\partial p_{1i} =$$

$$H_i(u,p_1, p_{M1}) + \sum_{1\leq j\leq n+1} p_{1j}[\partial H_j(u,p_1,p_{M1})/\partial p_{1i}] =$$

$$H_i(u,U(p_1,p_{M1}),p_1, p_{M1}) - \partial U(\cdot)/\partial p_{1i}, \qquad (32$$

since $\sum_{1\leq j\leq n+1} p_{1j}D_{ij} = 0$, and $\sum_{1\leq j\leq n+1} p_{1j}D_{(n+2)j} = D$, $i = 1,\ldots,n+1$.

From the preceding observations it follows that if I adopt H(·) and C(·), respectively, as the Hicksian demand function and the consumer's cost function, many of the restrictions that Deaton and Muellbauer put on their model's parameters are not restrictions on which my theory of consumer choice under uncertainty will insist.

3.6 Concluding remarks

To sum up for an empirical analysis of consumer choice under uncertainty. In the given theory of consumer choice under uncertainty the Marshallian Demand function has the adding-up property. However, it need not be homogeneous of degree zero in prices and net worth, and it does not satisfy the symmetry and negative semi-definite condition III. The compensated version; i.e., the Hicksian demand function, H(·), satisfies the equation,

$$F(p_1,p_{M1},A) = H(u, p_1, p_{M1}),$$

whenever $u = U(p_1,p_{M1}, F(p_1,p_{M1},A))$. However, it need not be homogeneous of degree zero in prices, and it need not be symmetric in the sense that, for all $u \in R_{++}$, $(p_1, p_{M1}) \in R_{++}^{n+1}$, and $i,j = 1, \ldots,n$,

$$\partial H_i(u,p_1,p_{M1})/\partial p_{1j} = \partial H_j(u, p_1,p_{M1})/\partial p_{1i} \text{ and}$$

$$\partial H_{n+1}(u,p_1,p_{M1})/\partial p_{1i} = \partial H_i(u,p_1,p_{M1})/\partial p_{M1}$$

Finally, the consumer's cost function, C(·), is an increasing function of u and satisfies the equation,

$$C(u, p_1, p_{M1}) = \sum_{1\leq i\leq n} p_{1i} H_i(u, p_1, p_{M1}) + p_{M1} H_{n+1}(u, p_1, p_{M1}).$$

However, it need not be a non-decreasing and concave function of (p_1, p_{M1}), and it need not be a linearly homogeneous function of (p_1, p_{M1}).

4. An Empirical Analysis of Consumer Choice under Uncertainty

In Stigum 2016 the author describes three different scenarios for empirical work in economics – one for applied econometrics in the tradition of Ragnar Frisch, another for applied econometrics in the tradition of Trygve Haavelmo, and a third for the kind of empirical analyses that are carried out by a large part of present-day econometricians. Frisch's researcher confronts his data with an axiomatized theory about a few undefined terms that live and function in an abstract model world. He uses bridge principles to describe how his theory variables are related to his data. Haavelmo's researcher, also, confronts his data with an axiomatized theory about the characteristics of variables that live in a model world. He identifies his theory variables with unobservable true data variables, and he uses error terms and auxiliary variables to relate the values of the true data variables to his own observed data variables. In the third scenario the researcher describes characteristics of his data variables by means of ad-hoc theoretical hypotheses. When he formulates his econometric model, he uses auxiliary variables and error terms to account for measurement errors and misspecification of equations.

In this section of the paper, I will generate data for consumer choice under uncertainty and use them and the prescriptions of applied econometrics in the tradition of Haavelmo to carry out an empirical analysis of an uncertainty version of Stone's Linear Expenditure System.

I imagine an empirical context in which 400 consumers, subject to the budget constraint each consumer faced, have purchased their preferred budget vectors – two commodities and a risky asset. The chosen vectors, presumably, maximized the values of the respective consumers' utility functions. I assume that the 400 utility functions were all models of the utility function in Example 1 with $a + b + c = 1$, and $\delta = \gamma = 0$; i.e., that

$$U(p_{11},p_{12},p_{13},x_1,x_2,\mu_1) = \alpha\log(x_1) + \beta\log(x_2) + E[B(P_2) + \log(\mu_1 - p_{21}d - p_{22}e) \mid P_1], \qquad (34)$$

where $P_1 = (p_{11},p_{12},p_{13})$, $P_2 = (p_{21},p_{22})$, and $B(P_2) = a\log(a/p_{21}) + b\log(b/p_{22}) + c\log(c/p_{23})$. In addition, I assume that the budget constraints that the consumers faced were all models of the equation,

$$p_{11}x_1 + p_{12}x_2 + p_{13}\mu_1 \leq A. \qquad (35)$$

Finally, I assume that the observations I have of the consumers' net worth, A, their choices of (x_1, x_2, μ_1), and the prices they faced, (p_{11}, p_{12}, p_{13}), constitute a random sample. I will use these observations to check if the uncertainty version of Stone's Linear Expenditure System is empirically relevant in the given empirical context.

I begin my econometric analysis by listing axioms for the uncertainty version of Stone's theory and axioms for the data generating process. Then I explain the need for axioms that delineate the relationship between true data variables and observed data variables. At last I present the statistical results and discuss their implications for the empirical relevance of Stone's Linear Expenditure System and for the salient characteristics of consumer choice under uncertainty.

4.1 Axioms for the uncertainty version of Stone's Theory

I imagine that the variables in Stone's theory belong in a theory universe. This theory universe is a triple, $(\Omega_T, \Gamma_T, (\Omega_T, \aleph_T, P_T(\cdot)))$, where Ω_T is a subset of a vector space, Γ_T is a finite set of assertions concerning properties of vectors in Ω_T, and $(\Omega_T, \aleph_T, P_T(\cdot))$ is a probability space. The latter comprises Ω_T, a σ field of subsets of Ω_T, $\aleph_T$, and a probability measure, $P_T(\cdot)): \aleph_T \rightarrow [0,1]$.

The assertions in Γ_T consist of five axioms, A 1-A 5.

A 1 $\Omega_T \subset R^3 \times R \times R^3 \times R^2$. Thus $\omega_T \in \Omega_T$ only if $\omega_T = (x,A,p_1,p_2)$ for some $x \in R^3$, $A \in R$, $p_1 \in R^3$, $p_2 \in R^2$, and $(x,A,p_1,p_2) \in R^9$.

A 2 $x \in R_+^3$, $A \in R_{++}$, $p_1 \in R_{++}^3$, and $p_2 \in R_{++}^2$.

In the intended interpretation of x, A, p_1, and p_2, $x = (x_1, x_2, \mu)$, where x_1 and x_2 are two different commodities and μ is a risky asset, A represents the net worth of the consumer, p_1 denotes the current-period price of x, and p_2 denotes the next-period price of (x_1, x_2).

Relative to $P_T(\cdot)$, the components of (x, A, p_1, p_2) are random variables. Axioms A 3 and A 4 delineate their characteristics.

A 3 Let $x(\cdot):\Omega_T \rightarrow R_+^3$, $A(\cdot):\Omega_T \rightarrow R_{++}$, $p_1(\cdot):\Omega_T \rightarrow R_{++}^3$, and $p_2(\cdot):\Omega_T \rightarrow R_{++}^2$ be defined by the equations,

$$(x(\omega_T), A(\omega_T), p_1(\omega_T), p_2(\omega_T)) = \omega_T \text{ and } \omega_T \in \Omega_T.$$

The vector-valued function, $(x,A,p_1,p_2)(\cdot)$, is measurable with respect to $\aleph_T$ and has, subject to the conditions on which Γ_T insists, a well-defined probability distribution relative to $P_T(\cdot)$. I refer to it as the RPD, where R is short for researcher, P for probability, and D for distribution.

A 4 Relative to $P_T(\cdot)$, the components of x, A, p_1, and p_2 have finite means and finite positive variances.

The last axiom, A 5, describes how the nine random variables, $x(\cdot)$, $A(\cdot)$, $p_1(\cdot)$, and $p_2(\cdot)$, are related to one another in the theory.

A5 There exist positive constants, α, β, e, and g, that satisfy the following three equations for all $\omega_T \in \Omega_T$:

$$x_1(\omega_T) = \alpha\cdot(p_{13}/p_{11})(\omega_T)\cdot E[(\mu(\omega_T) - p_{21}e - p_{22}g)^{-1} \mid P_1(\omega_T)\ ]^{-1}; \qquad (36)$$

$$x_2(\omega_T) = \beta\cdot(p_{13}/p_{12})(\omega_T)\cdot E[(\mu(\omega_T) - p_{21}e - p_{22}g)^{-1} \mid P_1(\omega_T)\ ]^{-1}; \qquad (37)$$

$$\mu(\omega_T) = [(A/p_{13})(\omega_T) - (\alpha+\beta)\cdot E[(\mu(\omega_T) - p_{21}e - p_{22}g)^{-1} \mid P_1(\omega_T)]^{-1}; \qquad (38)$$

and $p_{21}e + p_{22}g < \mu$,

where $E[(\cdot) \mid P_1(\omega_T)\ ]$ denotes the conditional expectation of $(\cdot)$ given the value of $P_1(\omega_T)$, and where d and g are taken to vary over sample consumers.

The five theory axioms have many models. In the intended interpretation of the fifth axiom, A5, the axiom records, for each $\omega_T \in \Omega_T$, the three necessary conditions for a maximum of Stone's expected-utility function that remain when one in Example 3 accounts for the budget constraint, substitutes $E[(a+b+c)(\mu_1 - p_{21}d - p_{22}e)^{-1} \mid P_1\ ]/\ p_{13}$ for λ, and takes $(a + b + c)$

to equal 1, and δ and γ to equal zero. The approximate solution to these necessary conditions together with error terms constitute the consumer's Marshallian demand function for (x_1, x_2, μ) in the theory.

The Marshallian demand function in the uncertainty version of Stone's Linear Expenditure System has the adding up property, and the equations in A 5 accord with that. The function is not homogeneous of degree zero in prices and net worth, and it is not symmetric in the way condition III in Section 2 prescribes. That was established in Example 4 above. However, the Marshallian demand function in Stone's Linear Expenditure System has other interesting properties that one can derive by implicit differentiation of the equations in A 5. Here are seven examples:

Let $E\mu P_1 = E[(\mu(\omega_T)-p_{21}e-p_{22}g)^{-1} \mid P_1(\omega_T)]^{-1}$, suppress ω_T, and observe that $E\mu P_1$ is an increasing positive function of μ. Then

$$\partial x_1/\partial p_{11} = -\alpha\cdot(p_{13}/p_{11}^2)\cdot E\mu P_1 + \alpha\cdot(p_{13}/p_{11})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{11} + \partial/\partial p_{11} EP_1] \quad (39)$$

$$\partial x_2/\partial p_{11} = \beta\cdot(p_{13}/p_{12})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{11} + \partial/\partial p_{11} EP_1] \quad (40)$$

$$\partial x_2/\partial p_{13} = \beta\cdot(1/p_{12})\cdot E\mu P_1 + \beta\cdot(p_{13}/p_{12})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{13} + \partial/\partial p_{13} EP_1] \quad (41)$$

$$(1 + (\alpha+\beta)\partial/\partial\mu E\mu P_1)\cdot\partial\mu/\partial p_{13} = [-(A/p_{13}^2) - (\alpha+\beta)\partial/\partial p_{13} EP_1] \quad (42)$$

$$(1 + (\alpha+\beta)\cdot\partial/\partial\mu E\mu P_1)\cdot\partial\mu/\partial p_{11} = -(\alpha+\beta)\partial/\partial p_{11} EP_1 \quad (43)$$

$$\partial x_1/\partial A = \alpha\cdot(p_{13}/p_{11})\cdot\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial A \quad (44)$$

$$(1 + (\alpha+\beta)\cdot\partial/\partial\mu E\mu P_1)\cdot\partial\mu/\partial A = (1/p_{13}) \quad (45)$$

These equations put restrictions on the parameters of the econometric model that I shall use in searching for the empirical relevance of the intended family of models of A 1 – A 5; e.g., $\partial\mu/\partial A > 0$, $\partial x_1/\partial A > 0$, and if $\partial/\partial p_{11} EP_1 < 0$ - as in

Example 5 below, then $\partial\mu/\partial p_{11} > 0$. Here and in equations (39) - (45) I have used EP_1 in the place of $E\mu P_1$ to indicate that the derivative, $\partial/\partial p_{11}EP_1$, is to be taken with respect to P_1 and not with respect to both μ and P_1.

The theory axioms may be hard to read. So here is an example to clarify the underlying ideas.

Example 5 Consider the necessary conditions for a maximum of the utility function in (34) subject to the condition in (35). These necessary conditions equal the equations in Axiom A 5 when ω_T is removed from each equation. Assume that d = e = 0.25, and that the prices of the two commodities and the risky asset satisfy the conditions:

$$p_{ij} € \{1, 2\}, i,j = 1,2; \text{ and } p_{13} = 1. \qquad (46)$$

In addition, assume that the conditional probabilities in the expression,

$$E[(\mu_1 - (0.25)\cdot(p_{21} + p_{22}))^{-1} \mid P_1], \qquad (47)$$

with $z = (p_{21} + p_{22})$, satisfy three conditions:

(1) When (p_{11}, p_{12}, p_{13})=(1,1.1),

(pr. z=2) is 2/3, (pr. z=3) is 1/6, (pr. z=4) is 1/6;

i.e., when P_1 = (1,1.1), the probability that $(p_{21} + p_{22})$ equals 2, 3, or 4 is, respectfully, (2/3), (1/6), and (1/6);

(2) when (p_{11}, p_{12}, p_{13}) = (1,2,1) or (2,1,1),

(pr. z = j) is 1/3, for j = 2, 3, and 4; and

(3) when (p_{11}, p_{12}, p_{13}) = (2,2.1),

(pr. z = 2) is 1/6, (pr. z = 3) is 1/6, (pr. z = 4) is 2/3.

Suppose that P_1 = (1,1,1). Then the three equations in A5 become

$$x_1 = \alpha[(2/3)(\mu_1-0.5)^{-1} + (1/6)(\mu_1-0.75)^{-1} + (1/6)(\mu_1 - 1)^{-1}]^{-1}$$
$$x_2 = \beta[(2/3)(\mu_1-0.5)^{-1} + (1/6)(\mu_1-0.75)^{-1} + (1/6)(\mu_1 - 1)^{-1}]^{-1}$$

$$\mu_1 = A - (\alpha + \beta) [(2/3)(\mu_1 - 0.5)^{-1} + (1/6)(\mu_1 - 0.75)^{-1} + (1/6)(\mu_1 - 1)^{-1}]^{-1}$$

With $\alpha = \beta = 0.5$, and A = 29.38, the vector, $(x_1, x_2, \mu_1) = (7.19, 7.19, 15)$, solves these equations.

Similarly, when $P_1 = (1,2,1)$ or $(2,1,1)$, the three equations in A5 can be written as

$$x_1 = (\alpha/\varphi)[(1/3)(\mu_1 - 0.5)^{-1} + (1/3)(\mu_1 - 0.75)^{-1} + (1/3)(\mu_1 - 1)^{-1}]^{-1}$$
$$x_2 = (\beta/\psi)[(1/3)(\mu_1 - 0.5)^{-1} + (1/3)(\mu_1 - 0.75)^{-1} + (1/63)(\mu_1 - 1)^{-1}]^{-1}$$
$$\mu_1 = A - (\alpha + \beta) [(1/3)(\mu_1 - 0.5)^{-1} + (1/3)(\mu_1 - 0.75)^{-1} + (1/3)(\mu_1 - 1)^{-1}]^{-1},$$

where φ denotes the pertinent value of p_{11} and ψ denotes the pertinent value of p_{12}. With $\alpha = \beta = 0.5$, and A = 29.250, the vector, $(x_1,x_2,\mu_1) = (7.125, 3.5625, 15)$ solves these equations when $P_1 = (1, 2, 1)$, and the vector, $(x_1, x_2, \mu_1) = (3.5625, 7.125, 15)$, solves the equations when $P_1 = (2, 1, 1)$.

Finally, when $P_1 = (2,2,1)$, the equations can be written as follows:

$$x_1 = (\alpha/2)[(1/6)(\mu_1 - 0.5)^{-1} + (1/6)(\mu_1 - 0.75)^{-1} + (2/3)(\mu_1 - 1)^{-1}]^{-1}$$
$$x_2 = (\beta/2)[(1/6)(\mu_1 - 0.5)^{-1} + (1/6)(\mu_1 - 0.75)^{-1} + (2/3)(\mu_1 - 1)^{-1}]^{-1}$$
$$\mu_1 = A - (\alpha + \beta) [(1/6)(\mu - 0.5)^{-1} + (1/6)(\mu - 0.75)^{-1} + (2/3)(\mu - 1)^{-1}]^{-1} .$$

With $\alpha = \beta = 0.5$, and A = 29.12, the vector, $(x_1, x_2, \mu_1) = (3.53, 3.53, 15)$, solves the equations.

In looking at the preceding triples of equations, it is interesting to observe that, when P_1 changes from (1,1,1) to (2,1.1), the consumer's demand for x_1 changes from 7.19 to 3.5625. This change is the sum of a substitution effect – due to the change in p_{11}, an expectations' effect – due to the change in the value of $E\{(\cdot) \mid P_1\}^{-1}$ from 14.38 to 14.25, and an income effect - due to the change in A from 29.38 to 29.25.

Similarly, when P_1 changes from (2,2.1) to (2,1,1), the consumer's demand for x_2 increases from 3.53 to 7.125. This change is the sum of a substitution effect – due to the change in p_{12}, an expectations' effect – due to the change in $E\{(\cdot) \mid P1\}^{-1}$ from 14.12 to 14.25, and an income effect - due to a change in A from 29.12 to 29. 25. These observations illustrate the kind of behavior characteristics which the equations in (39) – (45) depict.

4.2 Axioms for the Data Generating Process

I imagine that the data that I will use to test the empirical relevance of my theory axioms belong in a data universe. This data universe is a triple, $(\Omega_P, \Gamma_P, (\Omega_P, \aleph_P, P_P(\cdot)))$, where Ω_P is a subset of a vector space, Γ_P is a finite set of assertions concerning properties of vectors in Ω_P, and $(\Omega_P, \aleph_P, P_P(\cdot)))$ is a probability space. The latter comprises Ω_P, a σ field of subsets of Ω_P, $\aleph_P$, and a probability measure, $P_P(\cdot))$: $\aleph_P \rightarrow [0,1]$.

The assertions in Γ_P consist of four axioms, D 1- D 4. Here are the first two axioms.

D 1 $\Omega_P \subset R^3 \times R^3 \times R^2$. Thus, $\omega_P \in \Omega_P$ only if $\omega_P = (y, v, ma, v13)$ for some $y \in R^3$, $v \in R^3$, $(ma, v13) \in R^2$, and $(y, v, ma, v13) \in R^8$.

D 2 If $\omega_P \in \Omega_P$ and $\omega_P = (y, v, ma, v13)$ for some $(y, v, ma, v13) \in R^8$, then $v_1y_1 + v_2y_2 + v_3y_3 = ma$.

In the intended interpretation of these axioms, the eight components of the vectors in Ω_P denote so many units of two commodities, (y_1, y_2), a risky asset, y_3, the prices of y_1, y_2, and y_3, (v_1, v_2, v_3), and two auxiliary variables, ma and v13, that are taken to measure the initial net worth and the income expectations of a consumer. Whether my interpretation of y, v, ma, and v13 is empirically relevant in the empirical context that I described above, remains to see.

Relative to $P_P(\cdot)$, the components of y, v, ma, and v13 are random variables. Axioms D 3 and D 4 describe their characteristics.

D 3 Let $y(\cdot):\Omega_P \rightarrow R^3$, $v(\cdot):\Omega_P \rightarrow R^3$, $(ma, v13)(\cdot)) :\Omega_P \rightarrow R^2$ be defined by the equations,

$$(y(\omega_P), v(\omega_P), (ma, v13)(\omega_P)) = \omega_P \text{ and } \omega_P \in \Omega_P.$$

The vector-valued functions, $y(\cdot)$, $v(\cdot)$, and $(ma, v13)(\cdot)$ are measurable with respect to $\aleph_P$ and have, subject to the conditions on which Γ_P insists, a well-defined joint probability distribution, the TPD, where T is short for true, P for probability, and D for distribution.

D 4 Relative to $P_P(\cdot)$, $y(\cdot)$, $v(\cdot)$, and $(ma, v13)(\cdot)$ have finite means and finite positive variances.

In the intended interpretation of the assumptions in D1 – D 4, the TPD plays the role of the data generating process. I assume that TPD has one ‘true’ model, and that the data variables in this model have finite means and finite positive variances. The researcher does not know what the ‘true’ model of TPD looks like.

In the present theory-data confrontation the axioms of the data universe, Γ_P , delineate an identifiable and low-level-theory-consistent structure of the empirical analysis. The low-level axioms provide a framework within which the high-level theory – here the family of models of A 1 – A 5 - can be tested (cf. Bontemps and Mizon (2003) [5] (p. 365)).

To introduce the high-level theory into the present empirical analysis, I follow Haavelmo's suggestion (cf. Haavelmo 1944, pp. 7-8) and identify the values of the theory variables with the true values of the corresponding observed data variables. In addition, I identify the values of functions of theoretical variables with functions of the true values of pertinent data variables. That means in the present analysis that I will identify the values of the components of (x_1, x_2, μ) with the true values of the corresponding components of (y_1, y_2, y_3), and the values of (p_{11}, p_{12}, p_{13}) with the true values of the corresponding components of (v_1, v_2, v_3). Similarly, I will identify the value of A with the true value of ma, and I will take the true value of v13 to be a measure of the value of $E[(\mu_1 - p_{21}d - p_{22}e)^{-1} \,|\, P_1]$. Finally, I will identify the values of the three functions,

$$\alpha\cdot(p_{13}/p_{11})\cdot E[(\mu_1 - p_{21}d - p_{22}e)^{-1} \,|\, P_1]^{-1}, \qquad (48)$$

$$\beta\cdot(p_{13}/p_{12})\cdot E[(\mu_1 - p_{21}d - p_{22}e)^{-1} \,|\, P_1]^{-1}, \text{ and} \qquad (49)$$

$$((A/p_{13}) - (\alpha+\beta)\cdot E[(\mu_1 - p_{21}d - p_{22}e)^{-1} \,|\, P_1]^{-1}), \qquad (50)$$

respectively, with the true values of the linear functions,

$$(a_j\cdot v13 + b_j\cdot p_{11} + c_j\cdot p_{12} + d_j\cdot p_{13} + e_j\cdot ma),\ j = 4,5, \text{ and} \qquad (51)$$

$$(g_6 + a_6\cdot v13 + b_6\cdot p_{11} + c_6\cdot p_{12} + d_6\cdot p_{13} + e_6\cdot(ma/p_{13})). \qquad (52)$$

Here, it is important to observe that the equations in (51) and (52) are not meant to be linear approximations to the non- linear functions in (48)-(50). In the empirical analysis, the TPD value of a coefficient in equations (51)-(52) is taken to have the same sign as an estimate of the coefficient if it lies in the estimate's 95% confidence interval. In addition, I assume that in a 95% confidence region of the estimates of the means of the data variables in D 1 and D 5, the derivatives of the functions in equations (48)-(50) have the same signs as the true values – i.e., the TPD values – of the corresponding derivatives in equations (51)-(52).

With the given identification in mind, I can in two additional axioms, D 5 and D 6, formulate the data version of A 1 – A 5, and delineate the relationship between the true and the observed values of the data variables.

D 5 There exist a constant G, three five-tuples of constants, $(a_i, b_i, c_i, d_i, e_i)$, i = 1,2,3, and a triple of random variables, $(\eta_1, \eta_2, \eta_3)(\cdot)$, such that, for all $\omega_P \in \Omega_P$,

$$y_1(\omega_P)=a_1v13(\omega_P)+b_1v_1(\omega_P)\ c_1v_2(\omega_P)+d_1v_3(\omega_P)+e_1ma(\omega_P)+\eta_1(\omega_P); \tag{53}$$

$$y_2(\omega_P)=a_2v13(\omega_P)+b_2v_2(\omega_P)+c_2v_2(\omega_P)+d_2v_3(\omega_P)+e_2ma(\omega_P)+\eta_2(\omega_P); \tag{54}$$

$$y_3(\omega_P)=G+a_3v13(\omega_P)+b_3v_1(\omega_P)+c_3v_2(\omega_P)+d_3v_3(\omega_P)+e_3(ma/v_3)(\omega_P)+\eta_3(\omega_P); \tag{55}$$

The true values of the constants in (53) – (55) – i.e. their values in the TPD – are such that the true values of the data variables satisfy the three equations without the error terms.

D 6 In the TPD, $\eta_1(\cdot)$, $\eta_2(\cdot)$, and $\eta_3(\cdot)$ have finite means, finite positive variances, and may be orthogonal to the independent variables in equations (53) – (55). They may be correlated, but they need not be normally distributed.

4.3 The empirical relevance of the intended family of models of A 1 – A 5

I begin the empirical investigations by estimating the means of the data variables in D 1 and D 5, and - with Stata-16's structural equation modeling program - by estimating the values of the parameters in equations (53) – (55). The means are recorded in Table 1, and the ML estimates of the parameters in equations (53) – (55) are listed in Table 2.

Table 1

The means of the data variables in D 1 and D 5

The variables in D 1 are not normally distributed as Dornik-Hansen's multivariate normality test reveals: chi2(12) = 3824.518 Prob>chi2 = 0.0000. In addition, the error terms in the estimated equations are not normally distributed. According to the Dornik-Hansen multivariate normality test of the three error variables, chi2(6) = 1053.517 Prob>chi2 = 0.0000. Consequently, the accuracy of the 95% confidence intervals in the two tables and the t-values in Table 2 are suspect (cf. in this respect Davidson and MacKinnon, 1993, pp. 88-112). To justify my use of these measures, I must appeal to asymptotic theory and show that in the TPD the parameter estimators are consistent, and that the estimates are asymptotically normally distributed. The necessary arguments for that are standard and omitted here.

Table 2

Simultaneous ML estimates of the parameters in equations (53) – (55)

The error terms in the three estimated equations are correlated. However, for the intended purposes of Section 4, they are orthogonal to all the independent variables except v_3. The correlation matrix in Table 3 attests to that.

Table 3

Correlation matrix of independent data variables and the error terms

In applied econometrics in the tradition of Haavelmo, the empirical relevance of a theoretical hypothesis is determined the way a null hypothesis is tested in mathematical statistics. The theoretical hypothesis is taken to be empirically relevant in a given empirical context if and only if it cannot be rejected.

To see if his Linear Expenditure System was empirically relevant, Stone checked whether his econometric model satisfied the adding-up, the homogeneity, and the symmetry property of a Marshallian demand function. Unfortunately, I cannot use any of these characteristics in my search for the empirical relevance of the uncertainty version of Stone's System. The adding-up property is satisfied in Axiom A 5. My data, also, satisfy the adding-up property in the sense that $v_1 \cdot y_1 + v_2 \cdot y_2 + v_3 \cdot y_3 = ma$. However, the assumptions I make in D 5 and D 6 do not ensure that the true model of D 1 – D 6 has the adding-up property. The other two properties are irrelevant according to Example 4 above. That leaves me with the equations in (39) – (45) and their many analogues. Whether I can use them to test the empirical relevance of the uncertainty version of Stone's Linear Expenditure System remains to see.

To use equations (39 – (45) and their analogues in my search for the empirical relevance of the uncertainty version of Stone's System, I must look for contradictions and confirmations. Contradictions will cause rejection of Stone's System. If there are no contradictions, the confirmations will provide information about consumer behavior characteristics. In this respect three remarks are in order.

Remark 1 In the present case, the theory – i.e., the intended family of models of A 1 – A 5 – is about consumer choice under uncertainty. When I (1) identify theory variables with true values of data variables, (2) identify functions of theory variables with functions of data variables with true values, and (3) relate the probability distribution of the true values of the data variables to a data generating process – the TPD – that has only one model, the estimates in Table 2 become estimates of characteristics of a typical consumer's behavior under uncertainty.

Remark 2 My description of the behavior under uncertainty of a typical consumer tells how he will react to changes in the values of pertinent parameters; e.g., how his demand for x_1 will change when its price, p_{11}, changes. Note, therefore, that the description is about the signs of estimated parameters and not about the values of these parameters. The signs have a meaning only if the TPD values of the respective estimates belong in the parameters' confidence intervals, and the TPD values have the same signs as the estimated parameters.

Remark 3 The equations in (39) – (45) display examples of the results of differentiating the functions in (48) – (50) implicitly with respect to A and the components of P_1. In searching for confirmations, I associate the signs of derivatives of the functions in equations (51) – (52) with the signs of the corresponding derivatives of functions in (48) – (50), and I assume that the sign of a derivative in (51) - (52) is also the sign of the corresponding derivative in (48) – (50). I take a relation in (39) – (45) to be confirmed if the signs of the pertinent derivatives in (51) – (52) are meaningful and if the given relation makes economic-theoretic sense.

So far, I have failed to find contradictions. The confirmations I can establish may be controversial, but here they are:

B1. I associate the signs of $\partial y_1/\partial v_1$ and $\partial y_2/\partial v_2$, respectively, with the signs of

$$-\alpha\cdot(p_{13}/p_{11}^2)\cdot E\mu P_1 + \alpha\cdot(p_{13}/p_{11})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{11}+\partial/\partial p_{11}EP_1], \text{ and}$$

$$-\beta\cdot(p_{13}/p_{12}^2)\cdot E\mu P_1 + \beta\cdot(p_{13}/p_{12})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{12}+\partial/\partial p_{1a}EP_1]. \quad (56)$$

In addition, I associate the signs of $\partial y_1/\partial v_2$ and $\partial y_2/\partial v_1$, respectively, with the signs of

$$\alpha\cdot(p_{13}/p_{11})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{12}+\partial/\partial p_{12}EP_1] \text{ and}$$

$$\beta\cdot(p_{13}/p_{12})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{11}+\partial/\partial p_{11}EP_1]. \quad (57)$$

Finally, since the true value of v13 is taken to be a measure of the value of $E[(\mu_1 - p_{21}d - p_{22}e)^{-1} \mid P_1]$, I associate the signs of $\partial y_1/\partial v13\cdot$ and$\cdot\partial y2/\partial v13$, respectively, with the signs of

$$-\alpha\cdot(p_{13}/p_{11})\cdot(E\mu P_1)^2 \text{ and } -\beta\cdot(p_{13}/p_{12})\cdot(E\mu P_1)^2. \quad (58)$$

If I do, it follows from equations (39) and (40) that

$$\partial x_1/\partial p_1 < 0,\ \partial x_2/\partial p_{12} < 0, \text{ and that } \partial x_1/\partial p_{12} < 0 \text{ and } \partial x_2/\partial p_{11} < 0. \quad (59)$$

The last two inequalities imply that the two commodities are complements and not substitutes. The first two inequalities imply that demand for the respective commodities will increase when its price falls.

In this context it is interesting that the given inequalities are in accord with the consumer's response to price changes in Example 5. Their signs are, also, in accord with the signs of the corresponding derivatives in Stone's certainty theory.

Here it is important to recall that equation (43) and its analogue for $\partial\mu/\partial p_{12}$ insist that $\partial/\partial p_{11}EP_1$and $\partial\mu/\partial p_{11}$ have opposite signs and that $\partial/\partial p_{12}EP_1$and $\partial\mu/\partial p_{12}$ have opposite signs. The same equations, also, insist that if my assignment of signs to the two equations in (57) is correct, then it must be the case that ·

$$\partial\mu/\partial p_{11} > 0 \text{ and } \partial/\partial p_{12}EP_1 < 0, \text{ and that}$$

$$\partial\mu/\partial p_{12} > 0, \text{ and } \partial/\partial p_{12}EP_1 < 0. \qquad (60)$$

B2. I associate the signs of $\partial y_1/\partial v_3$ and $\partial y_2/\partial v_3$, respectively, with the signs of

$$\alpha\cdot(1/p_{11})\cdot E\mu P_1 + \alpha\cdot(p_{13}/p_{11})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{13}+\partial/\partial p_{13}EP_1] \text{ and}$$

$$\beta\cdot(1/p_{12})\cdot E\mu P_1 + \beta\cdot(p_{13}/p_{12})\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{13}+\partial/\partial p_{13}EP_1]. \qquad (61)$$

If I do, it follows from equation (41) and its analogue for $\partial x_2/\partial p_{13}$that

$$\partial x_1/\partial p_{13} > 0, \partial x_2/\partial p_{13} > 0. \qquad (62)$$

This makes sense even for models of the axioms with small negative values of $[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{13}+\partial/\partial p_{13}EP_1]$.

B3. In accord with the signs of the functions in (57), I associate the signs of $\partial y_3/\partial v_1$, $\partial y_3/\partial v_2$ with the signs of

$$(\alpha+\beta)\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{11} + \partial/\partial p_{11}EP_1],$$

$$(\alpha+\beta)\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{12} + \partial/\partial p_{12}EP_1]. \qquad (63)$$

Moreover, I associate the sign of $\partial y_3/\partial v_3$ with the sign of

$$-(A/p_{13}^2) - (\alpha+\beta)\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{13} + \partial/\partial p_{13}\, EP_1]. \qquad (64)$$

Finally, I associate the sign of $\partial y_3/\partial v13$ with the sign of

$$(\alpha+\beta)\cdot(E\mu P_1)^2. \qquad (65)$$

If I do, it follows from equation (43) and its analogue for $\partial\mu/\partial p_{12}$ that

$$\partial\mu/\partial p_{11} > 0 \text{ and } \partial\mu/\partial p_{12} > 0 - \qquad (66)$$

in accord with equation (60). Similarly, it follows from equation (42) that

$$\partial\mu/\partial p_{13} < 0. \qquad (67)$$

Remark 4 In B3 I associate $\partial y_3/\partial v_1$ and $\partial y_3/\partial v_2$, respectively, with $(\alpha+\beta)\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{11} + \partial/\partial p_{11}\, EP_1]$ and $(\alpha+\beta)\cdot[\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{12} + \partial/\partial p_{12}\, EP_1]$, and use equation (43) to observe that $\partial\mu/\partial p_{11} > 0$ and $\partial\mu/\partial p_{12} > 0$. Now, according to (43), $\partial\mu/\partial p_{11}$ and $\partial/\partial p_{11} EP_1$ have opposite signs, and the same must be true of $\partial\mu/\partial p_{12}$ and $\partial/\partial p_{12} EP_1$. Consequently, when $\partial/\partial p_{1j}\, EP_1 < 0$, $j = 1,2$, my assignment of signs to the given derivatives makes sense only for models of the axioms in which

$$\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial p_{1j} < -\,\partial/\partial p_{1j}\, EP_1,\ j = 1,2. \qquad (68)$$

B4. I associate the signs of $\partial y_1/\partial ma$, $\partial y_2/\partial ma$, and $\partial y_3/\partial(ma/v_3)$, respectively with the signs of

$$\alpha\cdot(p_{13}/p_{11})\cdot\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial A,\ \beta\cdot(p_{13}/p_{12})\cdot\partial/\partial\mu E\mu P_1\cdot\partial\mu/\partial A, \text{ and}$$
$$(1+(\alpha+\beta)\cdot\partial/\partial\mu E\mu P_1)^{-1}. \qquad (69)$$

If I do, it follows from equation (44) and its analogue for $\partial x_2/\partial A$ that

$$\partial x_1/\partial A > 0 \text{ and } \partial x_2/\partial A > 0, \qquad (70)$$

and it follows from equation (45) that

$$\partial\mu/\partial A > 0. \tag{71}$$

The signs of the first two derivatives are in accord with the signs of the corresponding derivatives in Stone's certainty theory. The arguments that underlie the last inequality follow from equation (45) and the relations, $p_{13}\cdot(\partial/\partial ma)(ma/p_{13}) = (\partial/\partial ma)ma$.

So far I have deliberated about variables and functions in an axiomatic system with five axioms for theory variables, A 1-A 5, and six axioms for data variables, D 1-D6. The theory axioms describe characteristics of consumer choice under uncertainty. The first four of the data axioms describe characteristics of a data generating process in which each observation records a consumer's choice of two commodities and one risky asset, the prices he faced, and his initial net worth. The last two of the data axioms, D 5-D 6, describe how theory variables and data variables are related to one another. With such a broad interpretation of the axioms in mind, I have in B1-B4 associated signs of derivatives of estimates of the functions in (53)-(55) with the signs of the corresponding derivatives of the functions in (48)-(50). From this I have inferred how a typical sample consumer in an empirically relevant model of A 1-A 5 would respond to changes in the prices he faced and in his initial net worth. Equations (59 and (70) record how his demand for commodities would change if his net worth and the commodities' prices changed, and equation (62) describes how his demand for commodities changes if the price of the risky asset changeS. Similarly, equations (65), (66), and (71) describe how the typical consumer's demand for the risky asset would respond to changes in his initial net worth and in the prices he faces.

With the given broad interpretation of the axioms, the theory axioms, A 1-A 5, have many very different models. One subfamily of these models constitutes the intended family of models of Example 1's uncertainty version of Stone's Linear Expenditure System. When I adopt this subfamily as the intended interpretation of A 1-A 5 and use it in my empirical analysis, my estimates of equations (52)-(55) yield new and interesting information about consumer choice under uncertainty. B5 attests to that.

B5 In the intended interpretation of A 1-A 5, the axioms describe consumer choice in an uncertainty version of Stone's Linear Expenditure System. In this theory, the typical consumer's responses to price changes and changes in his initial net worth that I record in equations, (59), (62), (65), (66), (70), and (71), can be decomposed into a substitution effect, an income effect, and an expectations' effect. According to Example 3, the signs of the three income effects equal the signs of $- x_1(D_{41}/D), - x_2(D_{42}/D), - \mu(D_{43}/D)$, where D and D_{4j}, j = 1,2,3, equal, respectively, the determinant and the 4jth cofactor of the matrix in Example 3. These three signs are negative since D > 0 and D4j > 0, j = 1,2,3. Now, the equations in (21) and their analogues for j = 1,2 imply that the signs of the three income effects equal the signs of $- x_1\partial x_1/\partial A$, $- x_2\partial x_2/\partial A$, $- \mu\partial\mu/\partial A$. From this and the equations, (70) and (71), I conclude that in the given sample the three income effects of a change in prices on the typical consumer's demand for commodities and a risky asset are negative. Similarly, in the theory the three expectations' effects equal $-D^{-1}\partial/\partial p_{11}E[(\mu_1-p_{21}d-p_{22}e)^{-1} \mid P_1]D_{31}$, $-D^{-1}\partial/\partial p_{12}E[(\mu_1-p_{21}d-p_{22}e)^{-1} \mid P_1]D_{32}$, and $-D^{-1}\partial/\partial p_{13}E[(\mu_1 - p_{21}d - p_{22}e)^{-1} \mid P_1]D_{33}$, where D and D_{3j}, j = 1,2,3, are, respectively, the determinant and three cofactors of the matrix in Example 3. Since $D > 0$, $D_{31} > 0$, $D_{32} > 0$ and $D_{33} < 0$, it follows that the two commodities' expectations' effects are, respectfully, positive constant multiples of $-\partial/\partial p_{11}EP_1$ and $-\partial/\partial p_{12}EP_1$, and the expectations' effect on the consumer's demand for μ is a negative multiple of $-\partial/\partial p_{13}EP_1$. Now, equations (43) and (66) imply that in the given sample the signs of the expectations' effects on a typical consumer' demand for commodities when their respective prices change are positive. This, then, accords with the expectations' effect in a model of the uncertainty version of Stone's theory in which $\partial/\partial p_{11}EP_1$ and $\partial/\partial p_{12}EP_1$ are negative. In the given sample, equations (42) and (67) imply that the expectations' effect on a typical consumer's demand for the risky asset when the price of μ changes is positive. This accords with the expectations' effect in a model of the uncertainty version of Stone's theory in which $-(\alpha+\beta)\partial/\partial p_{13}EP_1 < (A/p_{13}^2)$.

Concluding remarks

I have found no contradictions for rejecting the empirical relevance of the uncertainty version of Stone's Linear Expenditure System. Hence, according to my calculations, in the present empirical setting Stone's System is empirically relevant. This result is interesting if the econometric arguments I used are sound. I based my analysis on a system of necessary conditions for a constrained maximum of a consumer's utility function, differentiated these necessary conditions, and used the derivatives and data to describe how consumers react to changes in prices and net worth. The arguments are simple and they display an interplay between theory and data that is novel. So, if it is all right to base the empirical analysis of consumer choice on the necessary conditions themselves rather than on their solutions, my econometric arguments are sound.

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Table 1

The means of the data variables in D 1 and D 5

Variables	Mean	Std. Err.	[95% Conf.	Interval]
y_1	0.8989252	0.0232084	0.8532992	0.9445512
y_2	6.990683	0.0480411	6.896238	7.085128

y_3	4.987664	0.0522511	4.884942	5.090385
v_1	1.9	0.0150188	1.870474	1.929526
v_2	1.67	0.0235401	1.623722	1.716278
v_3	0.9500001	0 .0011432	0.9477526	0.9522476
ma	18.4457	0.2280978	17.99727	18.89412
v13	2.144075	0.0263041	2.092363	2.195787
ma/v_3	19.41889	0.239191	18.94866	19.88912

Table 2

Simultaneous ML estimates of the parameters in equations (53) – (55)

The estimates for y_1 in equation (53)

Variables	Coef.	Std. Err.	t	P>|t|	[95% Conf. Interval]	
v13	−0.2525318	0.0371883	−6.79	0.000	−0.3254196	−0.179644
v_1	−0.2197993	0.0620879	−3.54	0.000	−0.3414894	−0.0981092
v_2	−1.586188	0.1171962	−13.53	0.000	−1.815888	−1.356488
v_3	1.271565	0.1555858	8.17	0.000	0.9666219	1.576507
ma	0.178661	0.0123428	14.47	0.000	0.1544695	0.2028525
_cons	0 (constrained)					

The estimates for y_2 in equation (54)

Variables	Coef.	Std. Err.	t	P>|t|	[95% Conf. Interval]	

v13	−0.4088519	0.0325657	−12.55	0.000	−0.4726794	−0.3450243
v_1	−0.1592775	0.0543739	−2.93	0.003	−0.2658483	−0.0527066
v_2	−0.957302	0.1026002	−9.33	0.000	−1.158395	−0.7562092
v_3	4.913284	0.136256	36.06	0.000	4.646227	5.180341
ma	0.2767115	0.0108051	25.61	0.000	0.2555338	0.2978891
_cons	0 (constrained)					

The estimates for y_3 in equation (55)

Variables	Coef.	Std. Err.	t	P>|t|	[95% Conf. Interval]	
v13	1.943006	0.0030833	630.16	0.000	1.936962	1.949049
v_1	− 0.0127003	0.0027608	−4.60	0.000	−0.0181114	−0.0072891
v_2	−0.0873297	0.007359	−11.87	0.000	−0.101753	−0.0729064
v_3	−0.8827636	0.0544198	−16.22	0.000	−0.9894244	−0.776102
ma/v_3	0.0088881	0.0007386	12.03	0.000	0.0074404	0.010335
_cons	1.657825	0.0570665	29.05	0.000	1.545976	1.769673

Table 3

Correlation matrix of independent data variables and the error terms

Variable	η1	η2	η3	v13	v1	v2	v3	ma	ma/v_3
η1	1.0000								
η2	−0.9589	1.0000							

η3	−0.6379	0.7276	1.0000						
v13	−0.0373	0.0393	0.0288	1.00					
v1	−0.0580	0.0609	0.0447	-0.0365	1.0000				
v2	−0.0325	0.0342	0.0251	-0.0724	0.0319	1.0000			
v3	−0.3807	0.4003	0.2937	-0.6128	0.0000	0.0000	1.0000		
ma	−0.0351	0.0368	0.0256	0.1012	0.0894	0.9271	0.0729	1.0000	
ma/v_3	0.0068	−0.0019	0.0023	0.1602	0.0908	0.9303	−0.0206	0.9953	1.0000